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THE GENERALIZED RIEMANN PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS AND THE ALLIED PROBLEMS FOR LINEAR DIFFERENCE AND q-DIFFERENCE EQUATIONS.

By George D. Birkhoff.

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The program of obtaining a characterization of a function in simple descriptive terms which are independent of the equations of definition of the function is a familiar one. To Riemann is due the formulation of this characterization for the algebraic functions and for the functions defined by ordinary linear differential equations without irregular singular points. In both of these instances the characterization involves a certain number of characteristic constants — the monodromic group constants in the last mentioned instance. Riemann also proposed the associated problem of assigning these constants at pleasure.[1]

During the last few years I have discovered that the program admits of extension in a number of directions. The aim of the present paper is to solve the generalized problem of Riemann for ordinary linear differential equations with irregular singular points, and the analogous problem for linear difference equations and for linear q-difference equations. The formulation of the first and second of these problems has been given by me earlier.[2] At about the same time as myself, Nörlund, in his fundamental work on linear difference equations, was led to formulate essentially the second problem.[3] The third is stated in the present paper.

The problem of Riemann for linear differential equations in its

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1 Werke, (zweite Auflage) pp. 37–39, 67–69.

2 Trans. Am. Math. Soc., **10**, 436–470 (1909), and **12**, 243–284 (1911). These two papers will be referred to as I and II respectively.

3 Mémoires de l'Académie Royale des Sciences et des Lettres de Danemark, series 7, **6**, 309–326 (1911); C. R. vol. 156, pp. 200–202 (1913).

In the second of these papers, Nörlund gives a formulation and explicit solution of the hypergeometric difference equation problem.

classic form was first solved by Hilbert.[4] His treatment and Plemelj's elegant completion thereof[5] reposed alike upon a certain theorem whose proof was made by means of the Fredholm theory. Owing to the deep-seated analogy between linear differential and difference and q-difference equations, I have been able to apply a convenient extension of the same theorem in all cases; my proof is based on a method of successive approximations.

Inasmuch as I have been able to simplify Hilbert's and Plemelj's treatment of the classic Riemann problem, I have ventured to include my treatment of it also.

Part I. The Preliminary Theorem.

§ 1. *Some Definitions.*

Let C be a simply closed analytic curve in the complex x-plane. If the arc length along this curve from a fixed to a variable point is measured by s, and if l be the length of C, it is clear that x is a single-valued analytic function of s with period l for s real, and that dx/ds is not zero. Consequently if we introduce a new variable τ defined by

$$\tau = e^{\frac{2\pi\sqrt{-1}s}{l}},$$

a one-to-one analytic correspondence is set up between the points of the unit circle $|\tau| = 1$ in the τ-plane and the points of C. It will therefore be possible to choose $\rho > 1$ so that the circular ring in the τ-plane,

$$\frac{1}{\rho} \leqq |\tau| \leqq \rho,$$

is transformed in a one-to-one and conformal manner into a ring in the x-plane bounded by simply closed analytic curves C_1 and C_2, within and without C respectively, while at the same time the circle $|\tau| = 1$ is transformed into C. Let $\tau = \tau(x)$ be the function which effects this transformation.

Also let $a(x)$ be any function continuous together with its derivatives of all orders along C,[6] and analytic save at a finite number of

4 Gött. Nachr. (1905), pp. 307–338.

5 Monatsh. f. Math. u. Phys., **19**, 205–246 (1908).

6 By definition we take $df(x)/dx$ along a curve L as follows:

$$\frac{df(x)}{dx} = \lim_{x' \doteq x} \frac{f(x') - f(x)}{x' - x} \qquad (x',\ x \text{ on } L).$$

points of C. These restrictions on $a(x)$ ensure that we can choose regular curves D_1 and D_2, within and without C respectively, and osculating C at the points where $a(x)$ is not analytic (Fig. 1) in such a way that on the continua limited by D_1, D_2, we have

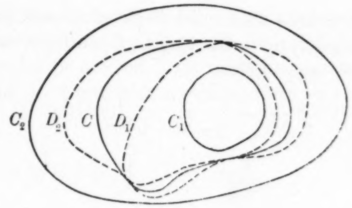

FIG. 1.

$$(1) \qquad |a(x)| \leqq K, \qquad \left| \frac{a(x) - a(x')}{x - x'} \right| \leqq K;$$

in these continua $a(x)$ is defined as the analytic extension of $a(x)$ on C. It is possible to extend further the definition of $a(x)$ throughout the ring formed by C_1, C_2 in such wise that inequalities of the type (1) hold; for this purpose it is clearly sufficient to choose real and imaginary components that join on continuously to the like components of $a(x)$ along D_1 and D_2, and to make each component satisfy inequalities of the same nature as (1). Such a choice can always be made.

§ 2. *On a First Type of Integral.*

Let us turn now to consider the integral

$$(2) \qquad \frac{1}{2\pi\sqrt{-1}} \int_C \frac{\tau^p(t)\, g^+(t)}{t - x}\, a(t)dt,$$

where $g^+(x)$ is a function analytic within C and continuous along C, and p is zero or a positive integer. Following Plemelj (loc. cit.) we shall term a function $g^+(x)$ of this description a *regular inner function* and affix to it a superscript $+$; likewise a superscript $-$ will indicate that a function is a *regular outer function,* i. e. is analytic in the extended plane without C, and continuous along C.

We can demonstrate at once that the integral (2) represents a regular inner function $f^+(x)$, or a regular outer function $f^-(x)$, according as x is within or without C. In the first place these functions are analytic within and without C respectively, as appears from (2).

In the second place, by Cauchy's integral theorem we have

$$(3) \quad 0 = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{\tau^p(t)\, g^+(t)}{t - x} dt - \frac{1}{2\pi\sqrt{-1}} \int_{C_1} \frac{\tau^p(t)\, g^+(t)}{t - x}\, dt,$$

provided that x lies between C and C_2, since the function $\tau^p(x)\, g^+(x)$ is analytic in the ring C, C_1 and continuous along its boundary. Thus we may write

$$(4)\quad f^-(x) = \frac{1}{2\pi\sqrt{-1}} \int_C \tau^p(t)\, g^+(t)\, \frac{a(t) - a(x)}{t - x}\, dt + \frac{a(x)}{2\pi\sqrt{-1}} \int_{C_1} \frac{\tau^p(t)\, g^+(t)}{t - x}\, dt.$$

The first integrand on the right-hand side is continuous in x and t, for t on C and x in the ring C, C_2 unless $x = t$, when the integrand is not defined; in the neighborhood of a pair of values $x = t$, the integrand remains finite by (1). Hence the first integral approaches a continuous limit as x approaches the boundary. Inasmuch as t is restricted to lie on C_1 in the second integrand, the same statement is certainly true of the second integral. Hence $f^-(x)$ may be so defined as to be continuous along C.

Likewise by means of the relation

$$(5)\quad \tau^p(x)\, g^+(x) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{\tau^p(t)\, g^+(t)}{t - x}\, dt - \frac{1}{2\pi\sqrt{-1}} \int_{C_1} \frac{\tau^p(t)\, g^+(t)}{t - x}\, dt,$$

valid for x between C and C_1 by Cauchy's integral formula, we obtain

$$(6)\quad f^+(x) - \tau^p(x) g^+(x)\, a(x) = \frac{1}{2\pi\sqrt{-1}} \int_C \tau^p(t)\, g^+(t)\, \frac{a(t) - a(x)}{t - x}\, dt + \frac{a(x)}{2\pi\sqrt{-1}} \int_{C_1} \frac{\tau^p(t)\, g^+(t)}{t - x}\, dt.$$

From this equation we can at once infer that $f^+(x)$ may be so defined as to be continuous along C.

Thus $f^+(x)$ and $f^-(x)$ are respectively regular inner and outer functions.

A comparison of the relations (4) and (6) which are both valid along C gives us the fundamental equation

$$(7)\qquad f^+(x) - f^-(x) = \tau^p(x)\, g^+(x)\, a(x) \qquad \text{along } C.$$

Let us now consider the maximum modulus of $f^-(x)$ outside of or along C. This maximum modulus, and likewise that for $g^+(x)$ within or along C, are attained on C of course. Suppose that we have along C

$$(8)\qquad |\, g^+(x)\, | \leqq L.$$

Now modify the contour C of integration in (4) to D_1. The integrand is analytic in t over the continua enclosed by C and D_1, so that the value of the integral will not thereby be altered. (It must be remembered that x lies without C in (4).) From this modified form of (4) we obtain

$$(9)\qquad |f^-(x)| \leqq \frac{KL}{2\pi}\left\{\int_{D_1} |\tau^p(t)\,dt| + \int_{C_1}\left|\frac{\tau^p(t)}{t-x}\,dt\right|\right\},$$

upon applying (8) and (1).

But the two integrals which appear in the right hand member of this inequality tend to zero as the unspecified integer p increases; in fact we have $|\tau(x)| < 1$ within C so that $\tau^p(x)$ tends uniformly to zero in any closed continuum within C, as p becomes infinite. It is to be observed that the quantity $|t-x|$ which appears in the second integrand is never less than the minimum distance from C to C_1, since x lies without C, and t is a point of C_1.

These considerations demonstrate that for a given positive ϵ, however small, the integer p may be chosen so large that *for every regular inner function* $g^+(x)$, we have

$$(10)\qquad \text{maximum of } |f^-(x)| \leqq \epsilon\,\{\text{maximum of } g^+(x)\} \quad \text{along } C.$$

§ 2. *On a Second Analogous Type of Integral.*

In the same way we may treat an integral

$$(2')\qquad \frac{1}{2\pi\sqrt{-1}}\int_C \frac{\tau^{-p}(t)\,g^-(t)}{t-x}\,a(t)\,dt,$$

where $g^-(x)$ is a regular outer function, and p is zero or a positive integer. As before we denote the value of the integral for x within C by $f^+(x)$, and for x without C by $f^-(x)$. A discussion parallel to the earlier one in § 1 shows that $f^+(x)$ and $f^-(x)$ as thus defined are regular inner and outer functions respectively; in this case equation (4) is replaced by

$$(4')\qquad f^+(x) = \frac{1}{2\pi\sqrt{-1}}\int_C \tau^{-p}(t)\,g^-(t)\,\frac{a(t)-a(x)}{t-x}\,dt + \frac{a(x)}{2\pi\sqrt{-1}}\int_{C_2}\frac{\tau^{-p}(t)\,g^-(t)}{t-x}\,dt,$$

and (6) likewise by

$$(6')\ f^-(x)+\tau^{-p}(x)\,g^-(x)a(x) = \frac{1}{2\pi\sqrt{-1}}\int_C \tau^{-p}(t)\,g^-(t)\,\frac{a(t)-a(x)}{t-x}\,dt + \frac{a(x)}{2\pi\sqrt{-1}}\int_{C_2}\frac{\tau^{-p}(t)\,g^-(t)}{t-x}\,dt.$$

From these two equations there results at once

$$(7')\qquad f^+(x) - f^-(x) = \tau^{-p}(x)\,g^-(x)\,a(x) \quad \text{along } C.$$

In order to develop an inequality for the modulus of $f^+(x)$ in this case, we note that the contour C in (4′) may be modified to D_2. The modification yields

$$(9')\qquad |f^+(x)| \leqq \frac{KL}{2\pi}\left\{\int_{D_2}|\tau^{-p}(t)\,dt| + \int_{C_2}\left|\frac{\tau^{-p}(x)}{t-x}\,dt\right|\right\}$$

where L is the maximum of $|g^-(x)|$ along C. But $|\tau^{-p}(x)|$ tends to zero for x outside of C as p becomes infinite, since for such an x we have $|\tau(x)| > 1$. We conclude therefore that for any positive ϵ however small, the integer p may be taken so large that *for every regular outer function* $g^-(x)$ we have

$$(10')\quad \text{maximum of } |f^+(x)| \leqq \epsilon\,\{\text{maximum of } |g^-(x)|\} \quad \text{along } C.$$

A further property of the function $f^-(x)$, which is apparent from its definition, is that this function vanishes at $x = \infty$.

§ 3. *Solution of a Pair of Matrix Equations.*

Throughout the present paper we shall be concerned with linear equations in n unknown functions, whose complete solution may be expressed in terms of n particular solutions. On this account we shall employ the matrix notation.

We consider first a pair of *matrix equations*

$$(11)\qquad \begin{cases} F^+(x) - F^-(x) = \tau^p(x)\,G^+(x)\,A(x), \\ G^-(x) - G^+(x) = \tau^{-p}(x)\,F^-(x)\,A^{-1}(x) - I, \end{cases} \qquad \text{along } C.$$

Here $\tau(x)$ is the function defined in § 1; the matrix $A(x)$ is a given matrix $(a_{ij}(x))$ $(i, j = 1, \ldots, n)$ of which each element is defined along C and has the properties specified in § 1 for the function $a(x)$ (namely, it is continuous together with its derivatives of all orders along C, and analytic save at a finite number of points); furthermore the determinant $|A(x)|$ is not to be zero along C. The symbol I stands

for the unit matrix (δ_{ij}) in which $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$, and $A^{-1}(x)$ stands for the matrix inverse to $A(x)$. The matrices $F^+(x)$, $G^+(x)$, $F^-(x)$, $G^-(x)$ are to be determined to satisfy (11), the first two as matrices of regular inner functions and the last two as matrices of regular outer functions. The matrix products on the right hand side are the customary matrix products, and the factors $\tau^p(x)$, $\tau^{-p}(x)$ stand for the matrices $(\tau^p(x)\,\delta_{ij})$ and $(\tau^{-p}(x)\,\delta_{ij})$ respectively.

It may be proved without difficulty that for p taken large enough a solution of these equations exists. To effect this proof we apply a method of successive approximations based on the sequence of equations

$$F_0^+(x) = F_0^-(x) = G_0^-(x) = O, \quad G_0^+(x) = I,$$

$$(12)\quad \begin{cases} F_1^+(x) - F_1^-(x) = \tau^p(x)\,G_0^+(x)\,A(x), \\ G_1^-(x) - G_1^+(x) = \tau^{-p}(x)\,F_0^-(x)\,A^{-1}(x) - I, \end{cases} \quad \text{along } C,$$

$$\begin{cases} F_2^+(x) - F_2^-(x) = \tau^p(x)\,G_1^+(x)\,A(x), \\ G_2^-(x) - G_2^+(x) = \tau^{-p}(x)\,F_1^-(x)\,A^{-1}(x) - I, \end{cases} \quad \text{along } C,$$

. . . .

The symbol O is used to denote a matrix of zero elements, and the superscripts $+$ and $-$ are used to designate matrices of regular inner and outer functions respectively.

If we write $P_0^-(x) = F_0^-(x) = O$, $Q_0^+(x) = G_0^+(x) = I$, and furthermore

$$(13)\quad \begin{cases} P_m^+(x) = F_m^+(x) - F_{m-1}^+(x), & P_m^-(x) = F_m^-(x) - F_{m-1}^-(x), \\ Q_m^+(x) = G_m^+(x) - G_{m-1}^+(x), & Q_m^-(x) = G_m^-(x) - G_{m-1}^-(x), \end{cases}$$

it is clear that the sequence of equations (12) is equivalent to

$$(14)\quad \begin{cases} P_m^+(x) - P_m^-(x) = \tau^p(x)\,Q_{m-1}^+(x)\,A(x), & \text{along } C, \\ Q_m^-(x) - Q_m^+(x) = \tau^{-p}(x)\,P_{m-1}^-(x)\,A^{-1}(x), & (m = 1, 2, \ldots). \end{cases}$$

Here the superscripts are employed as before.

The form of equations (14) is such that we can determine $P_m^+(x)$, $P_m^-(x)$, $Q_m^+(x)$, $Q_m^-(x)$ in terms of $P_{m-1}^-(x)$, $Q_{m-1}^+(x)$ so that the mth pair of equations (14) is satisfied. In fact the first one of the mth pair of matrix equations may be broken up into n^2 ordinary equations

$$p_{i,j,m}^+(x) - p_{i,j,m}^-(x) = \tau^p(x)\sum_{\lambda=1}^{n} q_{i,\lambda,m-1}^+(x)\,a_{\lambda j}(x) \quad \text{along } C,$$
$$(i, j = 1, \ldots, n),$$

where the third subscript on the functions corresponds to the subscript on the matrix. We have already obtained a solution of an equation of the form

$$f^+(x) - f^-(x) = \tau^p(x)\, q_{i,\lambda,m-1}^+(x)\, a_{\lambda j}(x) \quad \text{along } C$$

(compare with (7)) in the form of a definite integral. By forming the sum of $f^+(x)$ and $f^-(x)$ for $\lambda = 1,\ldots,n$ we obtain for every i and j, elements $p_{i,j,m}^+(x)$ and $p_{i,j,m}^-(x)$ which form the elements of $P_m^+(x)$ and $P_m^-(x)$ with the desired property (14).

Likewise we can break up the second one of the mth pair of matrix equations (14) into n^2 equations. A solution may here be built up in a similar way (compare (7′). It must be observed that since the determinant of $A(x)$ does not vanish along C, the elements of $A^{-1}(x)$ satisfy the conditions imposed on $a(x)$ at the outset.

Now if we recall the method of solution of (14), it is clear from (10), (10′) that along C the maximum modulus of any element of $P_m^-(x)$ or $Q_m^+(x)$ does not exceed $n\epsilon L_{m-1}$ where L_{m-1} denotes the maximum modulus of any element of $P_{m-1}^-(x)$ or $Q_{m-1}^+(x)$ along C, and ϵ is arbitrarily small uniformly for all values of m. This relation may be expressed in the simpler form

$$L_m \leqq n\epsilon L_{m-1}. \tag{15}$$

The series formed by the elements in any ith row and jth column of the series of matrices

$$P_o^-(x) + P_1^-(x) + \ldots, \qquad Q_o^+(x) + Q_1^+(x) + \ldots,$$

will therefore converge absolutely and uniformly provided that ϵ is taken so small that $n\epsilon < 1$.

But the sums of $m + 1$ terms of these two series of matrices are $F_m^-(x)$ and $G_m^+(x)$ respectively, whose elements therefore converge uniformly to the elements of matrices $F^-(x)$ and $G^+(x)$ of regular outer and inner functions respectively. If we recall that $P_o^-(x) \equiv O$, and that the integral form of representation of each element of $P_m^-(x)$ (see (2)) makes each element of this matrix reduce to zero at $x = \infty$, it is plain that at infinity $F^-(x)$ reduces to the matrix O.

If we turn now to consider $F_m^+(x)$ and $G_m^-(x)$ along C, we see from what precedes and from equations (12), that these matrices also converge uniformly along C, and therefore respectively within C and without C, to matrices $F^+(x)$ and $G^-(x)$ of regular outer and inner functions. Since $G_o^-(x) = O$, it is clear that $G^-(x)$, as well as $F^-(x)$, reduces to O at $x = \infty$.

The matrices $F^-(x)$, $F^+(x)$, $G^-(x)$, $G^+(x)$ thus obtained will satisfy (11), as appears from (12) by letting n become infinite.

§ 4. *Application to the Solution of a Single Matrix Equation.*

Multiply the second matrix equation (11) on the right by $\tau^p(x)A(x)$ and subtract it, member for member, from the first equation (11). There results

$$F^+(x) = \tau^p(x)[I + G^-(x)]A(x) \quad \text{along } C. \tag{16}$$

Inasmuch as $G^-(x)$ reduces to a matrix of zero elements at $x = \infty$, the determinant of $I + G^-(x)$ and also of $F^+(x)$ is not identically zero.

The matrix equation (16) admits of further simplification. In fact the function $\log \tau(x)$ is analytic along C and increases by $2\pi\sqrt{-1}$ as x makes a positive circuit of C. If c lies within C the function

$$\phi(x) = \log \tau(x) - \log(x-c)$$

is accordingly single-valued as well as analytic along C. But $\phi(x)$ is of the form

$$\tau^p(x)\, g^+(x)\, a(x) \qquad (p = 0,\ g(x) = 1,\ a(x) = \phi(x)),$$

so that by (7) we can find $\theta^+(x)$ and $\theta^-(x)$ such that

$$\theta^+(x) - \theta^-(x) = \phi(x) \quad \text{along } C;$$

this gives us

$$\tau^p(x) = (x-c)^p e^{p\theta^+(x)} e^{p\theta^-}(x).$$

Now let us write

$$\Phi(x) = e^{-p\theta^+(x)} F^+(x), \qquad \Psi(x) = (x-c)^p e^{p\theta^-(x)}[I + G^-(x)].$$

By these equations we define $\Phi(x)$ as a matrix of regular inner functions, and $\Psi(x)$ as a matrix of functions analytic outside of C except for a pole of order p at $x = \infty$, and continuous along C; the determinant of neither $\Phi(x)$ nor $\Psi(x)$ vanishes identically. Between $\Phi(x)$ and $\Psi(x)$, by (16), we have the matrix relation

$$\Phi(x) = \Psi(x)\, A(x) \quad \text{along } C. \tag{17}$$

It is this type of matrix equation which is important for the present paper.

§ 5. *Further Properties of $\Phi(x)$ and $\Psi(x)$.*

It is necessary for us to investigate further the nature of any solution $\Phi(x)$, $\Psi(x)$ of (17) in the neighborhood of the curve C. In the first place, it is to be observed that at points of C where all of the elements of $A(x)$ are analytic, the same is true of the elements of $\Phi(x)$ and $\Psi(x)$; in truth, the equation (17) shows us that analytic extension is possible across the curve at such points.

We shall prove that the elements of these matrices possess derivatives of all orders, continuous at all points of C.

Since the elements of $A(x)$ have line derivatives of all orders along C we may write, for t and y upon C,

$$(18)\quad A(t) = A(y) + (t-y)\frac{d}{dy}A(y) + \dots + \frac{(t-y)^k}{k!}\frac{d^k}{dy^k}A(y) + (t-y)^k B(t, y),$$

where the elements of $B(t, y)$ are continuous functions of t and y along C.[7] Also by Cauchy's integral formula in matrix form we have from (17) for x within C

$$\frac{d^{k-1}\Phi(x)}{dx^{k-1}} = \frac{(-1)^{k-1}(k-1)!}{2\pi\sqrt{-1}}\int_C \frac{\Psi(t)\, A(t)}{(t-x)^k}\,dt \qquad (k = 2, 3, \dots).$$

If we substitute the above expression for $A(t)$ in this last equation, we obtain a number of terms of the form (save for a constant multiplier)

$$(19)\qquad \int_C \frac{(t-y)^l\,\Psi(t)}{(t-x)^k}\,dt\,\frac{d^l A(y)}{dy^l} \qquad (l < k).$$

The integral is not altered in value if C is replaced by C_2 which lies outside of C. Therefore each of these terms represents a function analytic in x and continuous in y along C. There remains a single term not of the form (19), namely

$$\frac{(-1)^{k-1}(k-1)!}{2\pi\sqrt{-1}}\int_C \left(\frac{t-y}{t-x}\right)^k \Psi(t)\, B(t, y)\, dt.$$

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7 When a differentiation or integration sign appears before a matrix, it is understood to apply to each separate element of the matrix. The stated property of $B(t, y)$ comes at once from the explicit formula

$$B(t, y) = \frac{1}{k!}\int_y^t \left(\frac{z-y}{t-y}\right)^k \frac{d^{k+1}A(z)}{dz^{k+1}}\,dz.$$

If now x be made to approach a point x_0 of C, and if y be taken as the foot of the normal from x to C, this term approaches a limit which is continuous along C. In fact the factor $(t-y)^k/(t-x)^k$ remains finite for t along C, and approaches the limit 1 uniformly save in the vicinity of x_0. Thus if x (and y also of course) approaches x_0, $d^{k-1}\Phi(x)/dx^{k-1}$ approaches a limit along C continuous for an arbitrary x_0.

A similar proof shows that $\Psi(x)$ has derivatives of all orders, continuous outside of and along C. This proof is based on the fact that the elements of $A^{-1}(x)$ satisfy the restrictions placed on the function $a(x)$ in § 1.

The above results also lead to the conclusion that at any point γ of C at which one or more elements of $A(x)$ fails to be analytic, the elements of $\Phi(x)$ and $\Psi(x)$ admit of asymptotic expansion in a series in positive integral powers of $x-\gamma$. This is an immediate consequence of an expansion like (18) for $\Phi(x)$ or $\Psi(x)$ in which now t and y can be any points within or without C respectively, and $B(t, y)$ is continuous in t and y.[8]

§ 6. *A Normal Form for $\Phi(x)$ and $\Psi(x)$.*

By a series of simple normalizations it is always possible to obtain a solution $\Phi(x)$, $\Psi(x)$ of (17) such that $\Phi(x)$ is (as before) a matrix of regular inner functions, and $\Psi(x)$ is a matrix of function analytic, without C in the extended plane except for a possible pole at $x=\infty$, *and furthermore such that* $|\Phi(x)|$ *does not vanish within or on* C, *and* $|\Psi(x)|$ *does not vanish without* C.

This solution may be directly obtained from that found in § 4. If $|\Phi(x)|$ vanishes at $x=c$ within C say, we can determine a matrix M of constants such that all the elements of the first row of $M\Phi(x)$ vanish at $x=c$, while $|M|\neq 0$. Now $M\Phi(x)$, $M\Psi(x)$ yield a new solution of (17), which has the properties given for $\Phi(x)$, $\Psi(x)$ in § § 4, 5. If we divide the elements of the first row of $M\Phi(x)$ by $x-c$, we obtain a matrix $\overline{\Phi}(x)$ of functions analytic within C and continuous along C; if the same operation be applied to $\Psi(x)$, we obtain $\overline{\Psi}(x)$, a matrix of functions analytic without C in the extended plane save for a possible pole at $x=\infty$. Moreover $\overline{\Phi}(x)$ and $\overline{\Psi}(x)$ yield a solution of (17), since the effect of this operation is to alter the matrix on either side only in the removal of a factor $x-c$ from the first row. By this device we have diminished the multiplicity of the zero of $|\Phi(x)|$ at

[8] For the relation between the existence of derivatives and of asymptotic series see W. B. Ford, Bull. Soc. Math. France, **39**, 347–352 (1911).

$x = c$ by one unit, without the introduction of further zeros of $|\Phi(x)|$ or $|\Psi(x)|$ in the finite plane.

An entirely similar process eliminates a zero of $|\Psi(x)|$ without C, or a zero of $|\Phi(x)|$ and $|\Psi(x)|$ along C. In consequence of the results of § 5, if either of these functions vanishes along C, the other does also, both at least to the first order.

This process may be continued so long as there remain zeros of $\Phi(x)$ or $\Psi(x)$. It must however finally come to an end. If this were not the fact it would follow at once that both $|\Phi(x)|$ and $|\Psi(x)|$ have a zero of infinite multiplicity at some one point of C where an element of $A(x)$ fails to be analytic. But this cannot be the case, for let $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ be a solution of the following matrix equation

$$\tilde{\Phi}(x) = A^{-1}(x)\,\tilde{\Psi}(x) \quad \text{along } C, \tag{20}$$

where the elements of $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ are restricted like $\Phi(x)$ and $\Psi(x)$ were found to be in § 4. The existence of such a solution becomes manifest by a mere interchange of the rôle of rows and columns in what precedes. Now from (17) and (20) we conclude that

$$|\Phi(x)| \cdot |\tilde{\Phi}(x)| = |\Psi(x)| \cdot |\tilde{\Psi}(x)| \quad \text{along } C.$$

The function represented by either side of this equality is not identically zero; and it appears from the two representations that it is analytic in the finite plane, and analytic or with a pole at infinity. Hence this function is a polynomial, and the multiplicity of the zeros of either $|\Phi(x)|$ or $|\Psi(x)|$ at any point of C is finite.

When the process comes to an end the following result has been obtained: if the elements of $A(x)$ are unlimitedly differentiable along C, analytic save at a finite number of points of C, and if $|A(x)|$ is not zero along C, there exists a solution $\Phi(x)$, $\Psi(x)$ of the equation (17)

$$\Phi(x) = \Psi(x)\,A(x) \quad \text{along } C,$$

in which the elements of $\Phi(x)$ are analytic within C, unlimitedly differentiable along C, and $\Phi(x)$ is of determinant not zero within or on C; and in which the elements of $\Psi(x)$ are analytic without C in the extended plane save for a possible pole at $x = \infty$, unlimitedly differentiable along C, and $\Psi(x)$ is of determinant not zero without C.

Here the point $x = \infty$ appears as an exceptional point. It is evident that an arbitrary point $x = \alpha$ not on C may be used to take the rôle of the point at infinity. In fact α may also be taken to be a point of C. When this is the case, the elements of $\Phi(x)$ and $\Psi(x)$ are finite,

or become infinite to finite order at $x = a$. To obtain such a solution $\Phi(x)$, $\Psi(x)$ it is only necessary to divide each element of $\Phi(x)$, $\Psi(x)$ by a suitable power of $x - a$, so chosen as to make each element of these matrices analytic at $x = \infty$; and then to apply the normalization above indicated, letting $1/x - a$ replace x. The curve C can also be taken to be a simply closed analytic curve which passes through $x = \infty$.

The main part of the conclusion that has been deduced above was obtained by Hilbert and Plemelj (loc. cit.) with the aid of the Fredholm theory of linear integral equations. Independently of their work, I treated a special case (see I, § 1) which arose in a different form in connection with my study of the irregular singular points of ordinary linear differential equations.

My proof in this special case suggested to me the above treatment of the general case by the method of successive approximations. The restrictions here placed on the elements of $A(x)$ and on the curve C are not essential to this treatment, and I have very little doubt that these may be replaced by the weaker restrictions of Hilbert and Plemelj. Nevertheless I have been content to use the simplest restrictions consistent with the applications in view.

A second proof in the special case has recently been given by me, Math. Ann., vol. 74 (1913) pp. 122 (see also Bull. Am. Math. Soc., vol. 18, 1911, p. 64). This second proof, which suggested itself to me at about the same time as the first, is practically the same as that given by Hilbert and Plemelj. To my considerable regret this relationship escaped my observation until it was too late for me to make suitable reference.

§ 7. *The Preliminary Theorem.*

The following is an extension of the preceding results which is convenient for the applications:

Preliminary Theorem. *Let $C_1, \ldots, C_r$ be r simply closed analytic curves in the extended complex plane. Let $A_1(x), \ldots, A_r(x)$ be matrices of functions defined and unlimitedly differentiable along $C_1, \ldots, C_r$ respectively, analytic save at a finite number of points of these curves and of determinant not zero. If furthermore at any point of intersection of C_α, C_β, the matrices $A_\alpha(x)$, $A_\beta(x)$ are such that the formal derivatives of all orders of the matrix*

$$A_\alpha(x)\, A_\beta(x) - A_\beta(x)\, A_\alpha(x) \tag{21}$$

vanish, there exists a matrix $\Phi(x)$ *with the following properties:*

(1) *each element of* $\Phi(x)$ *is analytic except along* $C_1, \ldots, C_r$ *and at an arbitrary point* $x = a$ *where the elements may become infinite to finite order;* $|\Phi(x)|$ *nowhere vanishes save possibly at* $x = a$*;*

(2) *the elements of* $\Phi(x)$ *are continuous and unlimitedly differentiable along each curve* C_i *from either side, analytic from either side save at points of intersection of the curves, or at those points where an element of* $A_i(x)$ *fails to be analytic, or at* $x = a$*; if* a *lies on a curve* C_i*, the matrix* $(x - a)^l A_i(x)$ [*or* $x^{-l} A_i(x)$ *if* $a = \infty$] *is unlimitedly differentiable along* C_i *for a suitable* l.[9]

(3) *if a + and − side of each curve* C_i *are chosen, then*

$$\lim_{x \doteq x_i^+} \Phi(x) = [\lim_{x \doteq x_i^-} \Phi(x)]\, A_i(x_i) \qquad (i = 1, \ldots, r),$$

where the approach to the arbitrary point x_i *of* C_i *is along the + and − side respectively.*

Let us begin by establishing the theorem in the case when a is not a point of $C_1, \ldots, C_r$. It has already been established for $r = 1$ (see § 6), with the single notational difference that $\Phi(x)$ was replaced by either of two matrices $\Phi(x)$ and $\Psi(x)$, according as x was within or without C. To establish the theorem then, we need only show that if it is true for $r \leqq k$ it is also true for $r = k + 1$, when the theorem follows by induction.

Assume that $\Phi_k(x)$ is the solution for $r = k$, and for the matrices $C_1, \ldots, C_k, A_1(x), \ldots, A_k(x)$, where $C_1, \ldots, C_{k+1}$, and $A_1(x), \ldots, A_{k+1}(x)$ satisfy the requirements of the theorem for $r = k + 1$. Let us suppose for the moment that a solution $\Phi_{k+1}(x)$ exists with the desired properties. If we write

$$\Phi_{k+1}(x) = U(x)\, \Phi_k(x), \tag{22}$$

the following facts are clear from (1), (2), (3) of the theorem: (1′) each element $U(x)$ is analytic except along $C_1, \ldots, C_{k+1}$ and at the specified point a, where its elements may become infinite to finite order; $|U(x)|$ nowhere vanishes save possibly at $x = a$; (2′) the elements of $U(x)$ are continuous and unlimitedly differentiable along each curve C_i from either side, analytic save at points of intersection

---

9 A function will be termed unlimitedly differentiable at $x = \infty$ if when we write $x = 1/x'$, the function of x' obtained by the substitution is unlimitedly differentiable at $x' = 0$.

of the curves or at those points where an element of $A_i(x)$ fails to be analytic;

$$
(3') \quad \begin{aligned} &\lim_{x \doteq x_i^+} U(x) = \lim_{x \doteq x_i^-} U(x) \qquad (i = 1, \ldots, k), \\ &\lim_{x \doteq x_{k+1}^+} U(x)\,\Phi_k(x) = [\lim_{x \doteq x_{k+1}^-} U(x)\,\Phi_k(x)]\,A_{k+1}(x_{k+1}). \end{aligned}
$$

The condition (3′) necessitates that $U(x)$ is analytic at any point of C_i $(i = 1, \ldots, k)$ and throughout the extended plane save at a and at points of C_{k+1}. The condition (3′) gives us in addition

$$
\lim_{x \doteq x_{k+1}^+} U(x) = [\lim_{x \doteq x_{k+1}^-} U(x)]\,\Phi_k(x_{k+1})\,A_{k+1}(x_{k+1})\,\Phi_k^{-1}(x_{k+1}).
$$

Conversely if $U(x)$ satisfies these conditions (1′), (2′), (3′) it is apparent that $\Phi_{k+1}(x)$, given by (22), will satisfy the conditions prescribed for $\Phi(x)$ in the theorem.

But in view of the above simplification of (3′), the conditions (1′), (2′), (3′) on $U(x)$ are precisely those of the theorem on $\Phi(x)$ if we take $r = 1$, $C = C_{k+1}$, $A_1(x) = \Phi_k(x)\,A_{k+1}(x)\,\Phi_k^{-1}(x)$. To complete a proof we need only show that this matrix fulfils the conditions prescribed in the theorem along C_{k+1}; for then a matrix $U(x)$ will exist which satisfies these conditions.

The elements of the matrix

$$
(23) \qquad \Phi_k(x)\,A_{k+1}(x)\,\Phi_k^{-1}(x)
$$

are analytic at points of C_{k+1} which are not points of intersection or contact with $C_1, \ldots, C_k$, or points at which an element of $A_{k+1}(x)$ fails to be analytic; at these latter points the elements of $\Phi_k(x)$ are analytic so that the above matrix is unlimitedly differentiable in the neighborhood of such a point. The determinant of this matrix is equal to $|A_{k+1}(x)|$ and nowhere vanishes along C_{k+1}.

It remains only to examine the elements of the matrix near points of intersection or contact of C_{k+1} with one of the curves $C_1, \ldots, C_k$. Suppose first that C_{k+1} intersects a single curve C_i at such a point. As x moves along the curve C_{k+1} and passes from the positive to the negative side of C_i, the matrix $\Phi_k(x)$ changes to $\Phi_k(x)\,A_i(x)$, so that the matrix (23) changes to

$$
\Phi_k(x)\,A_i(x)\,A_{k+1}(x)\,A_i^{-1}(x)\,\Phi_k^{-1}(x).
$$

Bearing in mind the condition of permutability imposed on $A_1(x), \ldots, A_r(x)$ at such a point of intersection (see (21)), it becomes apparent that the elements of the matrix (23) are continuous at this point, and have equal backward and forward derivatives of all orders

at the point. Thus the elements of (23) are unlimitedly differentiable in the neighborhood of such a point along C_{k+1}. The same is true at more complicated points of intersection or points of contact, as a similar argument shows.

The theorem is now demonstrated, at least for the case when a does not lie on a curve $C_1, \ldots, C_r$. If a does lie on such a curve, we first choose an a' which is not on these curves to replace a, and then resort to the simple device used at the close of § 6 to replace a' by a.

Part II: The Problem of Riemann and its Generalization.

§ 8. *On Cauchy Matrices.*

Let T be a matrix of constants of determinant not zero, corresponding to the coefficients of a linear transformation. According to the well-known theory of classification of such matrices, based on the Cayley-Weierstrass elementary divisor theory, we may write

$$T = C^{-1} I' C,$$

where C is a matrix of constants, and *in general* I' is of the form $(\rho_j \delta_{ij})$.

Consider along with the matrix I', the matrix of functions $I'(x) = (x^{k_j} \delta_{ij})$ where $2\pi k_j \sqrt{-1} = \log \rho_j$ $(j = 1, \ldots, n)$. When x makes a positive circuit of $x = 0$, $I'(x)$ becomes $I'(x) I'$. The matrix

$$T(x) = I'(x) C$$

will accordingly alter to

$$I'(x) I' C = I(x) C \cdot C^{-1} I' C = T(x) T,$$

when x makes such a circuit, i. e. $T(x)$ will be affected by the prescribed linear transformation.

Such a matrix $T(x)$ or a simple modification thereof exists for every transformation T and is called a Cauchy matrix.[10] A certain measure of arbitrariness enters into the determination of $T(x)$ when T is given; in general $k_1, \ldots, k_n$ are undetermined up to an additive integer. We note that the determinant of $T(x)$ is not zero for $x \neq 0, \infty$.

A final property of the Cauchy matrices which is important for us is that they form the matrix solution of a linear differential system

$$x \frac{dT(x)}{dx} = L T(x),$$

10 Cf. Schlesinger, Vorlesungen über lineare Differentialgleichungen, pp. 122–140, where a complete treatment is given.

where L is a matrix of constants. In fact we have

$$x \frac{dT(x)}{dx} = x \frac{dI'(x)}{dx} C.$$

But the matrix $x\, dI'(x) / dx$ may be written as

$$KI'(x), \qquad K = (k_j \delta_{ij}).$$

Thus we obtain

$$x \frac{dT(x)}{dx} = KI'(x)\, C = KC \cdot C^{-1} I'(x)\, C = LT(x).$$

§ 9. *The Monodromic Group Problem.*

The most elementary existence theorems for ordinary linear differential equations show that the linear differential system

$$\frac{dY(x)}{dx} = R(x)\, Y(x) \tag{24}$$

admits a matrix solution $Y(x)$ whose elements are analytic in the finite plane at every *ordinary point* where the elements of $R(x)$ are analytic, and analytic at infinity if the elements of $R(x)$ vanish to at least the second order at the ordinary point infinity; furthermore $|\, Y(x) \,|$ is not zero at such points. All other points of the plane are called *singular points* in contradistinction to the ordinary points. A finite singular point at which the elements of $R(x)$ are analytic or have a pole of the first order, or the point $x = \infty$ if each element of $R(x)$ vanishes to the first but not always to the second order at that point, is termed a *regular singular point.*

Suppose now that the elements of $R(x)$ are rational, and that all of the singular points $a_1, \ldots, a_m$ are regular. These will be taken to lie in the finite plane. It is easy to show that the elements of $Y(x)$ become infinite to only a finite order at $x = a_1, \ldots, a_m$.[11] When x makes a positive circuit of one of these points, $Y(x)$ changes to $Y(x)\, T_i$, where T_i is a matrix of constants and $|\, T_i \,| \neq 0$; in fact the most general solution is of the form $Y(x)\, T$ where T is an arbitrary matrix of constants for which $|\, T \,| \neq 0$, and after the circuit is made in the x-plane, $Y(x)$ is still a solution of (24).

If we start from a point $x = c$ and make a circuit of $a_1, \ldots, a_m$ in

[11] Cf. Trans. Am. Math. Soc., **11**, 199–202 (1910).

such wise that the combined circuit is reducible to a point, it is clear that

$$T_m T_{m-1} \ldots T_1 = I, \tag{25}$$

a necessary relation between the matrices $T_1, \ldots, T_m$.

The problem of Riemann is the following: For assigned points $a_1, \ldots, a_m$ and assigned matrices $T_1, \ldots, T_m$ for which (25) holds, to construct a matrix $Y(x)$ of functions of determinant not identically zero, with elements analytic save at $a_1, \ldots, a_m$ where these elements are finite or become infinite to finite order, and undergoing a transformation to $Y(x)\ T_i$ as x makes a positive circuit of $a_i\,(i = 1, \ldots, m)$.

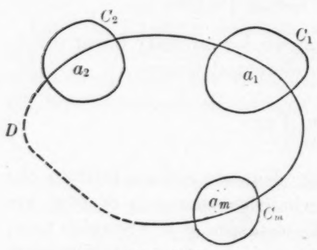

FIG. 2.

A solution of this problem has been given by Hilbert and completed by Plemelj (loc. cit.). It is possible to obtain a more simple solution on the basis of the preliminary theorem.

Let us surround $a_1, \ldots, a_m$ by small non-overlapping simply closed analytic curves $C_1, \ldots, C_m$ and let us pass through $a_1, \ldots, a_m$ in cyclical order another closed analytic curve D which meets each curve C_i only twice (Fig. 2), in l_i and m_i say.

Choose matrices $A_1 \ldots, A_m$, $A_{m+1} = A_1$ of constants such that

$$A_{i+1}^{-1} A_i = T_i \qquad (i = 1, \ldots, m). \tag{26}$$

Here A_1, for example, may be taken at pleasure and $A_2, \ldots, A_m$ are then determined. Define a matrix $\bar{A}(x)$ along D to be equal to A_i on that part of the curve which lies between C_i and C_{i+1} $[C_{m+1} = C_1]$ and equal to

$$\frac{x - m_i}{l_i - m_i} A_{i-1} + \frac{x - l_i}{m_i - l_i} A_i \qquad (i = 1, \ldots, m) \tag{27}$$

for x on the part of D within C_i. We will suppose that l_i, m_i and D were so chosen that D does not pass through one of the finite number of point for which the determinant of any matrix (27) vanishes.

The matrix $\bar{A}(x)$ as here defined is continuous along D, analytic save at the points where D intersects $C_1, \ldots, C_m$, and of determinant not zero along D.

By slightly modifying each element of $\bar{A}(x)$ along a small segment near either end of the arc of D within C_i for $i = 1,\ldots, m$, it is possible to obtain a matrix $A(x)$ whose elements are analytic save at the end points of these segments, and unlimitedly differentiable along D. Such a matrix $A(x)$ satisfies all of the conditions necessary for the application of the preliminary theorem in the case $r = 1$.

Taking D as the curve and $A(x)$ as the matrix, we can affirm the existence of a certain matrix $\Phi(x)$, possessing in particular the property

$$\lim_{x \doteq x_1^+} \Phi(x) = [\lim_{x \doteq x_1^-} \Phi(x)] A(x_1), \tag{28}$$

the approach to the point x_1 of D being from within and without D respectively. Let us choose $a = a_m$.

If we extend $\Phi(x)$ analytically across D between C_i and C_{i+1}, it becomes $\Phi(x)A_i$ by (28). If we extend this matrix analytically back across D between C_{i+1} and C_{i+2} $[C_{m+2} = C_2]$, it becomes $[\Phi(x)A_{i+1}^{-1}]A_i^{-1}$, or $\Phi(x)T_i$. That is, the matrix $U(x)$ obtained from $\Phi(x)$ by analytic extension is analytic outside of $C_1,\ldots,C_m$ and undergoes a transformation to $U(x)T_i$ when x makes a positive circuit of a_i. Furthermore the determinant of $U(x)$ is not zero outside of $C_1,\ldots,C_m$.

Denote by $Z_i(x-a_i)$ the Cauchy matrix belonging to the transformation T_i, so that $Z_i(x-a_i)$ undergoes a transformation to $Z_i(x-a_i)T_i$ as x makes a positive circuit of a_i. Write

$$Y(x) = Z(x)U(x), \tag{29}$$

where $Y(x)$ is the solution of the Riemann problem to be constructed.

The elements of $Z(x)$ must in the first place be single-valued and analytic outside of $C_1,\ldots, C_m$, since $U(x)$ undergoes the same transformation as that prescribed for $Y(x)$ about the points $a_1,\ldots,a_m$, and $|U(x)| \neq 0$.

Furthermore within C_i, the elements of $Y(x)$ are to be analytic except at a_i where they may become infinite to finite order. Hence the elements of the matrix $Y(x)Z_i^{-1}(x-a_i)$ must be single-valued and analytic within C_i, by the definition of $Z_i(x-a_i)$, except for a possible pole at $x = a_i$. Along C_i this matrix may be written

$$\bar{Z}_i(x) = Z(x)[U(x)Z_i^{-1}(x-a_i)] \qquad (i = 1, 2, \ldots m). \tag{30}$$

If we write $\Phi(x) = Z(x)$ outside of $C_1,\ldots,C_m$ and also $\Phi(x) = \bar{Z}_i(x)$ within C for $i = 1,\ldots,m$, the equations (30) may be written

$$(31) \quad \lim_{x=x_i^+} \Phi(x) = [\lim_{x=x_i^-} \Phi(x)] A_i(x_i), \qquad A_i(x) = U(x) Z_i^{-1}(x-a_i).$$
$$(i = 1, \ldots, m).$$

This suggests another application of the preliminary theorem, since the curves $C_1, \ldots, C_m$ and the matrices $A_1(x), \ldots, A_m(x)$ of known functions satisfy the necessary restrictions.

Let $\Phi(x)$ be the solution given by the theorem for $a = a_m$, and let $\overline{Z}_1(x), \ldots, \overline{Z}_m(x)$, $Z(x)$, be defined as equal to $\Phi(x)$ within $C_1, \ldots, C_m$ and outside of these curves respectively. These functions will then satisfy (30). Let $Y(x)$ be defined by (29). This matrix is clearly composed of elements analytic without and along $C_1, \ldots, C_m$, as are those of $U(x)$; within C_i, $Y(x)$ continues analytically into $\overline{Z}_i(x) Z_i(x - a_i)$ by (30) for $i = 1, \ldots, m$, and consequently its elements are analytic throughout the plane except possibly at $a_1, \ldots, a_m$; its elements become infinite only to a finite order at a_i since the elements of $Z_i(x - a_i)$ become infinite only to finite order at a_i. Furthermore by (29) $Y(x)$ undergoes a linear transformation to $Y(x) T_i$ as x makes a positive circuit of a_i. Thus the Riemann problem has been solved.

It is worthy of note that $|Y(x)|$ does not vanish for $x \neq a_i$ $(i = 1, \ldots, m)$. This is an immediate consequence of the fact that $|\overline{Z}_1(x)|, \ldots, |\overline{Z}_m(x)|$, $|Z(x)|$ do not vanish in their regions of definitions save at these points, and of the fact that the Cauchy matrix $Z_i(x - a_i)$ has a determinant which does not vanish save possibly at $x = a_i$ and $x = \infty$.

§ 10. *A Generalization. Equivalence.*

A more general result can be deduced exactly as the results of § 9 were obtained. Let us say that two matrices of functions $Y_1(x)$ and $Y_2(x)$ whose elements are analytic in the vicinity of $x = a$, but not in general single-valued or analytic at $x = a$, are *properly equivalent at* $x = a$ if we have

$$Y_1(x) = A(x) Y_2(x).$$

where $A(x)$ is composed of elements single-valued and analytic at $x = a$, of determinant not zero there; if this condition is not satisfied, but if the elements of $A(x)$ have a pole or are analytic at $x = a$, let us say that $Y_1(x)$ and $Y_2(x)$ are *improperly equivalent at* $x = a$.

This definition is convenient for the statement of the following result: *Let* $a_1, \ldots, a_m$ *be m given points; let* $T_1, \ldots, T_m$ *be matrices of constants such that* $T_m T_{m-1} \ldots T_1 = I$; *let* $Z_1(x), \ldots, Z_m(x)$ *be matrices of*

functions analytic of determinant not zero in the vicinity of a_i *and undergoing a transformation to* $Z_1(x)\ T_1, \ldots, Z_m(x)\ T_m$ *as* x *makes a positive circuit of* $a_1, \ldots, a_m$ *respectively. There exists then a matrix* $Y(x)$ *of functions of determinant not zero for* $x \neq a_1, \ldots, a_m$ *and analytic save at these points, which undergoes a transformation to* $Y(x)T_i$ as x *makes a positive circuit of* a_i $(i = 1, \ldots m)$; *furthermore* $Y(x)$ *is properly equivalent to* $Z_1(x), \ldots, Z_{m-1}(x)$ at $a_1, \ldots, a_{m-1}$ *and properly or improperly equivalent to* $Z_m(x)$ *at* a_m.

The result above stated is obtained when we replace the Cauchy matrices $Z_1(x-a_1), \ldots, Z_m(x-a_m)$ of § 9 by matrices $Z_1(x), \ldots, Z_m(x)$ having the properties specified. The line of attack is identical with that given in § 9. The facts concerning equivalence follow at once from the relations analogous to (30):

$$Y(x) = \overline{Z}_i(x)\, Z_i(x) \qquad (i = 1, \ldots, m). \tag{32}$$

Here $\overline{Z}_i(x)$ is composed of elements analytic at $x = a_i$ $(i = 1, \ldots, m-1)$, and of determinant not zero there; also $\overline{Z}_m(x)$ is composed of elements analytic at $x = a_m$ or with a pole at that point.

§ 11. *Final Form of Solution.*

There is a certain lack of symmetry between the rôle of $a_1, \ldots, a_m$ in the solution of the Riemann problem obtained in § 9, provided the given Cauchy matrices $Z_1(x-a_1), \ldots, Z_m(x-a_m)$ were taken in the most general possible form. We shall now proceed to show that *if* $Z_m(x-a_m)$ *be a properly chosen Cauchy matrix associated with the transformation* T_m, *the equivalence of* $Y(x)$ *and* $Z_m(x)$ *can be made proper at* $x = a_m$ *also.*

A form of reduction that is well-known suffices to establish this fact.[12] Nevertheless, inasmuch as a similar type of reduction is necessary later in the present paper, I give this reduction herewith.

Let us begin with the matrix $Y(x)$, obtained in § 9, which we may assume to be improperly equivalent at a_m to the Cauchy matrix $Z_m(x-a_m)$. Now we have (§ 8)

$$Z_m(x-a_m) = I'(x-a_m)\, C$$

so that from (30)

$$Y(x)\, C^{-1} = \overline{Z}_m(x)\, I'(x-a_m),$$

---

12 Plemelj, loc. cit., pp. 237–240.

The matrix of elements on the right can be written in the form

$$(x-a_m)^{k_1}(a_{11}+b_{11}(x-a_m)+\ldots),\ldots,(x-a_m)^{k_n}(a_{1n}+b_{1n}(x-a_m)+\ldots)$$

$$(33)\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(x-a_m)^{k_1}(a_{n1}+b_{n1}(x-a_m)+\ldots),\ldots,(x-a_m)^{k_n}(a_{nn}+b_{nn}(x-a_m)+\ldots)$$

Here it is supposed that the exponents of the Cauchy matrix do not differ by integers; but a similar form can be found in all cases. In each column of $Y(x)$ the highest possible power of $x - a_m$ is exhibited which leaves the coefficients of this power analytic in character at $x = a_m$.

If $|a_{ij}| \neq 0$ we can write this matrix in the form $A(x)I'(x - a_m)$, where $I'(x)$ is the matrix $(x^{k_j}\delta_{ij})$, and where $A(x)$ is the matrix obtained from (33) by striking out the exhibited powers of $x - a$. From the equation

$$Y(x) = A(x)\, I'(x - a_m)\, C^{-1}$$

we see that $Y(x)$ is properly equivalent to a Cauchy matrix belonging to T at $x = a_m$.

On the other hand if $|a_{ij}| = 0$ we proceed as follows: It is readily verified that if $Y(x)$ be one matrix solution of the Riemann problem satisfying the relation of equivalence given in § 10, then $DY(x)$ $[|D| \neq 0]$ is also a solution. Consequently it is no restriction in the consideration of (33) to assume that

$$a_{11} = a_{12} \ldots = a_{1n} = 0,$$

since when $|a_{ij}| = 0$ this relation may always be made to hold by multiplying on the left by a suitable matrix D. Denote by $y_1(x),\ldots,y_n(x)$ the elements of the first row after a factor $(x - a_m)^l$ has been removed, where l is the exponent of the highest power of $x - a_m$ that may be taken out and leave the elements $y_1(x),\ldots,y_n(x)$ of the respective forms

$$(x-a_m)^{k_1}(f_1+g_1(x-a_m)+\ldots),\ldots,(x-a_m)^{k_n}(f_n+g_n(x-a_m)+\ldots)$$

It follows that $f_1,\ldots,f_n$ are not all zero.

We may now add constant multiples of $y_1(x),\ldots,y_n(x)$ to the successive rows of $Y(x)$ and obtain a new matrix $Y(x)$. In fact this is equivalent to multiplying $Y(x)$ on the left by a matrix

$$P(x) = \left| \begin{array}{cccc} 1, & 0, & \ldots\ldots\ldots\ldots & 0 \\ \dfrac{c_2}{(x-a_m)^l}, & 1, & \ldots\ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \dfrac{c_n}{(x-a_m)^l}, & 0, & \ldots\ldots\ldots\ldots & 1 \end{array} \right|$$

Inasmuch as $| P(x) | = 1$, and the elements of $P(x)$ are analytic save at $x = a_m$, this modification cannot affect any of the properties already secured for $Y(x)$. However by choosing $c_2, \ldots, c_m$ properly we can clearly make all the coefficients $a_{2i}, \ldots a_{ni}$ vanish at $x = a_m$ provided $f_i \neq 0$. In this manner we can increase an exponent k_i by 1 without altering the determinant of $Y(x)$ except by a constant factor.

Inasmuch as $| Y(x) |$ does not vanish identically, a succession of steps of this type will finally bring to light a solution $Y(x)$ for which $Y(x)\, C$ has the form (33) and in addition $| a_{ij} | \neq 0$. When this stage is reached $Y(x)$ will be properly equivalent to a Cauchy matrix belonging to T_m at the point a_m.

When $Y(x)$ has thus been given a normal form, it is the solution of a linear differential system (24) with regular singular points at $x = a_1, \ldots, a_m$ and having no other singular points, as may be at once proved.[13] The elements of $R(x)$ therefore have the form of rational functions whose numerators are polynomials in x of degree at most $m - 2$, and whose denominators are the product of $(x - a_1), \ldots, (x - a_m)$.

§ 12. *Irregular Singular Points and Canonical Systems.*

The Cauchy matrix is the simplest possible matrix of functions to which a matrix solution of a given linear differential system is properly equivalent at a regular singular point. Let us determine the simplest possible matrix $Z(x)$ to which the matrix solution $Y(x)$ of a given linear differential system (24), in which the elements of $R(x)$ need not be rational, is properly equivalent at a prescribed *irregular* singular point. It is convenient to take this point to lie at infinity. If the highest order of any pole of an element of $xR(x)$ at $x = \infty$ is $p + 1$ $(p \geqq 0)$, then $p + 1$ is said to be the *rank* of the singular point $x = \infty$.

According to the results of § 10 we can find a matrix $Z(x)$ which at $x = \infty$ is properly equivalent to $Y(x)$ and at another point $x = 0$

[13] Schlesinger, loc. cit. pp. 215–221.

is improperly equivalent to a Cauchy matrix which at $x = 0$ undergoes a transformation inverse to that which $Y(x)$ undergoes at $x = \infty$. Here we take $m = 2$, $a_1 = 0$, $a_2 = \infty$. The condition $T_2 T_1 = I$ is satisfied.

By means of a modification precisely like that of § 11 we can make $Z(x)$ *properly* equivalent to a suitable Cauchy matrix at $x = 0$, and yet preserve the other properties listed in § 10.

Now consider

$$R_1(x) = \frac{dZ(x)}{dx} Z^{-1}(x).$$

Since $|Z|(x)| \neq 0$ for $x \neq 0, \infty$, and since $Z(x)$ and $dZ(x)/dx$ undergo the same substitution about $x = 0$, the elements of $R_1(x)$ are single-valued, and analytic for $x \neq 0, \infty$. Since $Z(x)$ is properly equivalent to a Cauchy matrix at $x = 0$, the elements of $R_1(x)$ have poles of at most the first order at $x = 0$.[14] Moreover since $Z(x)$ is properly equivalent to $Y(x)$ at $x = \infty$ we have

$$Z(x) = A(x)\, Y(x),$$

where the elements of $A(x)$ are analytic at $x = \infty$ and also $|A(x)| \neq 0$ at $x = \infty$. Therefore we obtain

$$\frac{dZ(x)}{dx} = A(x)\frac{dY(x)}{dx} + \frac{dA}{dx} Y(x) = \left[A(x)\, R(x) + \frac{dA}{dx}\right] Y(x),$$

and

$$R_1(x) = \frac{dZ(x)}{dx} Z^{-1}(x) = \left[A(x)\, R(x) + \frac{dA(x)}{dx}\right] A^{-1}(x).$$

Hence the elements of $R_1(x)$ are analytic or have a pole of order not greater than p at $x = \infty$.

From this analysis it follows that $xR_1(x)$ is a matrix of polynomials of degree at most $p + 1$, so that $Z(x)$ is itself the solution of a linear differential system

$$x\frac{dZ}{dx} = P(x)\, Z, \tag{34}$$

where $P(x)$ is a matrix of polynomials of degree at most $p + 1$. This is the *canonical* form of equation with an irregular singular point of rank $p + 1$ at $x = \infty$.

At a finite singular point $x = a$, the canonical system is of a type obtained from (34) by a transformation $x' - a = 1/x$.

14 The detailed proof is entirely similar to that given herewith to determine the nature of the elements of $R(x)$ at $x = \infty$. Cf. Schlesinger, loc. cit., pp. 143–144.

In order not to introduce artificial difficulties we shall consider the regular singular point to be of rank zero. To this case the above argument applies also, and the canonical system is that satisfied by a Cauchy matrix.

If $Y(x)$ is the matrix solution of a differential system (24) in which the elements of $R(x)$ are rational, the above argument shows that at each of its singular points $a_1, \ldots, a_m$, $Y(x)$ is properly equivalent to the matrix solutions $Z_1(x), \ldots, Z_m(x - a_m)$ of canonical differential systems.

Conversely the results of § 10 lead to the conclusion that given $T_1, \ldots, T_m$ such that $T_m T_{m-1} \ldots T_1 = I$, and canonical differential systems belonging to the singular points $a_1, \ldots, a_m$ with matrix solutions $Z_1(x), \ldots, Z_m(x)$ undergoing a transformation to $Z_1(x) T_1, \ldots, Z_m(x) T_m$ at these points respectively, there will exist a matrix solution $Y(x)$ of a rational differential system (24) which undergoes a transformation to $Y(x) T_i$ as x makes a positive circuit of a_i and which is properly equivalent to $Z_i(x)$ at this point, for $i = 1, \ldots, m - 1$, and properly or improperly equivalent to $Z_m(x)$ at $x = a_m$.

It is therefore essential to obtain a characterization of the matrix solution of a canonical system (34), and further to solve the associated inverse problem, before solving the general problem of characterization for a system (24) with irregular singular points.

§ 13. *The Problem of the Irregular Singular Point.*

In my paper referred to[15] I characterized the nature of the matrix solution of a canonical linear differential system (34), at least in the case that the roots of a certain characteristic equation were distinct; the case of equal roots introduces complications of an algebraical nature, and is put to one side in the present paper.

The results which I obtained may be recapitulated as follows: If the singular point is taken at $x = \infty$, there exists a formal matrix solution of (34)

$$S(x) = (e^{p_j(x)} x^{k_j} s_{ij}(x)),$$

$$(35) \qquad p_j(x) = \alpha_j \frac{x^{p+1}}{p+1} + \beta_j \frac{x^p}{p} + \ldots + \lambda_j x, \qquad (j = 1, \ldots, n),$$

$$s_{ij}(x) = s_{ij} + s_{ij}^{(1)} \frac{1}{x} + \ldots, \qquad (i, j = 1, \ldots, n),$$

[15] I, §§ 6, 7.

where $|s_{ij}| \neq 0$. The quantities $a_1, \ldots, a_n$ are the roots of the characteristic equation alluded to, and the series $s_{ij}(x)$ are in general divergent. We shall assume for the time being that no three of the points $a_1, \ldots, a_n$ lie on the same straight line in the complex plane. Let now $\tau_1, \ldots, \tau_N$ denote the $N = n(n-1)(p+1)$ arguments in order of increasing angular magnitude such that for some j and k $(j \neq k)$

$$(36) \qquad \Re\{(a_j - a_k)\, x^{p+1}\} = 0, \quad \arg x = \tau_m.$$

Here "$\Re$" denotes "the real part of". Let us write $\tau_{N+1} = \tau_1 + 2\pi$, and let j_m and k_m denote the value of j and k corresponding to m, so ordered that the real part (36) changes from positive to negative as $\arg x$ increases through $\arg x = \tau_m$. There exist then N matrix solutions $Z_1(x), \ldots, Z_N(x)$ such that for $i = 1, \ldots, N$

$$(37) \qquad Z_m(x) \sim S(x), \quad \tau_m \leqq \arg x < \tau_{m+1},$$ [16]

and such that along $\arg x = \tau_{m+1}$, $Z_{m+1}(x)$ and $Z_m(x)$ differ only in their j_mth column, the j_mth column of $Z_{m+1}(x)$ being obtained from that of $Z_m(x)$ by the addition of the k_mth column of $Z_m(x)$, affected with a suitable constant multiplier c_m, to the j_mth column. As a matter of definition we take

$$(38) \qquad Z_{N+1}(x) = Z_1(x)\, I', \quad I' = (e^{2\pi k_j \sqrt{-1}}\,\delta_{ij}).$$

The proof of the existence of $Z_1(x), \ldots, Z_N(x)$ having these properties can be directly based on the existence of a solution $Z(x)$ asymptotically represented by $S(x)$ along every particular ray.[17]

The properties so far stated are characteristic of the behavior of the matrix solution not alone of a canonical system but of any system with singular point of rank p at $x = \infty$ in the neighborhood of the singular point,[18] and are invariant under a transformation $\bar{Y}(x) = A(x)\, Y(x)$ where $A(x)$ is a matrix of elements analytic at the singular point in question and of determinant not zero there.

When $Z_1(x), \ldots, Z_{N+1}(x)$ are in addition the solution of a canonical system, the matrices $Z_m(x)$ are analytic in the finite plane for $x \neq 0$

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16 The relation "$z(x) \sim s(x)$, $\arg x = \sigma$," means in the present paper that $z(x)$ is asymptotically represented by $s(x)$ *in some sector (however small) that includes* $\arg x = \sigma$ *as an interior ray.* This slight modification of the conventional meaning of the symbol "$\sim$" and its natural extension to matrices is convenient for the present paper.

17 I, § 6.

18 I, § 6.

and of determinant not zero. At $x = 0$ these matrices are properly equivalent to a Cauchy matrix. These further facts come at once from the form of (34).

Conversely let us prove that if $Z_1(x), \ldots, Z_N(x)$ exist possessing such properties, they are all matrix solutions of one and the same canonical system (34).[19]

In the first place it is clear that the matrix

$$P(x) = x \frac{dZ_m(x)}{dx} Z_m^{-1}(x) \qquad (m = 1, \ldots, N)$$

is defined in each sector $\tau_m \leqq \arg x < \tau_{m+1}$ $(i = 1, \ldots, N)$ and hence is defined in the entire plane. Now $Z_{m+1}(x)$ is obtained from $Z_m(x)$ by multiplication on the right by the matrix

$$I + C_m,$$

where C_m is a matrix of zero elements except for the single element c_m in the k_mth row and j_mth column. It appears therefore that $P(x)$ is single-valued and analytic along each ray $\arg x = \tau_m$; for we have along this ray

$$\frac{dZ_{m+1}(x)}{dx} Z_{m+1}^{-1}(x) = \frac{dZ_m(x)}{dx}[I + C_m][I + C_m]^{-1}Z_m^{-1}(x)$$
$$= \frac{dZ_m(x)}{dx} Z_m^{-1}(x).$$

Moreover, on account of the equivalence of $Z_1(x), \ldots, Z_m(x)$ to a Cauchy matrix at $x = 0$, the elements of $xP(x)$ are analytic at $x = 0$.

Secondly, in the neighborhood of $x = \infty$, the matrix $P(x)$ is asymptotic to

$$x \frac{dS(x)}{dx} S^{-1}(x), \qquad \tau_m \leqq \arg x < \tau_{m+1}, \tag{39}$$

where the meaning of the notation is manifest. It is to be recalled that the relation (37) holds (by definition) in a small sector including $\arg x = \tau_m$ as an interior ray. This enables us to write

$$\frac{dZ_m(x)}{dx} \sim \frac{dS(x)}{dx}, \qquad \tau_m \leqq \arg x < \tau_{m+1}.^{20}$$

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19 I stated this fact without proof earlier; see I, p. 468.

20 Ford, loc. cit.

Thus $P(x)$ is represented asymptotically in the *complete* vicinity of $x = \infty$ by (39). But we have formally

$$\frac{dS(x)}{dx} = \left(e^{p_j(x)}x^{r_j}\left\{\left(\frac{dp_j(x)}{dx} + \frac{r_j}{x}\right)s_{ij}(x) + \frac{ds_{ij}(x)}{dx}\right\}\right). \tag{40}$$

If (35) and (40) be used in evaluating the expression (39), it is seen that each element of $P(x)$ is given asymptotically by a power series in descending integral powers of x, with leading term in x^{p+1} or lower power of x. It follows that the elements of $P(x)$ are analytic or have a pole of at most the $(p+1)$th order at $x = \infty$. Hence the elements of $P(x)$ are polynomials of degree at most $p + 1$.

The central problem of the irregular singular point is, for a given choice of $p_1(x), \ldots, p_n(x)$, $r_1, \ldots, r_n$ and of $c_1, \ldots, c_N$ above described, to construct a matrix $Y(x)$ with the above specified properties.[21]

§ 14. *Solution of the Problem of* § 13.

In order to solve the problem just stated we make an application of the preliminary theorem of Part I, taking $r = N/2$ and for the curves $C_1, \ldots, C_{\frac{1}{2}N}$ *not the* $N/2$ *straight lines formed by the* N *rays* $\arg x = \tau_m$ $(m = 1, \ldots, N)$ but by the N rays $\arg x = \tau'_m$ $(m = 1, \ldots, N)$, which are obtained from them by a slight rotation ϵ in a clockwise direction in the complex x-plane. It is evident that if ϵ be taken small enough we shall have

$$Z_m(x) \backsim S(x), \qquad \tau'_m \leqq \arg x \leqq \tau'_{m+1},$$

for $m = 1, \ldots, N$. Furthermore we shall have for any m

$$\Re\,(a_{j_m} - a_{k_m})\,x^{p+1} > 0, \qquad \arg x = \tau'_m,$$

for $m = 1, \ldots, N$. This fact is essential to the solution.

The matrices $A_1(x), \ldots, A_{\frac{1}{2}N}(x)$ which are to be used in the application of the theorem are defined as follows. Write

$$T(x) = (x^{p_j(x)}x^{r_j}\delta_{ij}),$$

and then put

$$\bar{A}_m(x) = T(x)\{I + C_m\}T^{-1}(x) \qquad (m = 1, \ldots, N), \tag{41}$$

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21 Cf. I, § 7.

where x in $\bar{A}_m(x)$ is taken along arg $x = \tau'_m$, and the determinations of $T(x)$ chosen are obtained from one another by allowing arg x to increase from τ_1 to τ_N. The matrix $\bar{A}_m(x)$ thus defined is analytic along its line save at $x = 0$. Also $\bar{A}_m(x) - I$ is a matrix whose only non-zero element is

$$c_m e^{p_{k_m}(x) - p_{j_m}(x)} x^{r_{k_m} - r_{j_m}}.$$

Now we have

$$p_{k_m}(x) - p_{j_m}(x) = (\alpha_{k_m} - \alpha_{j_m}) \frac{x^{p+1}}{p+1} + (\beta_{k_m} - \beta_{j_m}) \frac{x^p}{p} + \ldots + (\lambda_{k_m} - \lambda_{j_m})x,$$

which quantity by definition of j_m, k_m has *a negative real part* for x along the ray arg $x = \tau'_m$, at least when $|x|$ is sufficiently large. In consequence of the form of this non-zero element it is certain that $\bar{A}_m(x)$ is unlimitedly differentiable along the ray at $x = \infty$ and behaves there like the matrix I. As this is true along each of the rays, the matrices $\bar{A}_m(x)$ clearly satisfy the conditions of the theorem in the vicinity of $x = \infty$. Choose $A_1(x), \ldots, A_{\frac{1}{2}N}(x)$ along the straight lines $C_1, \ldots, C_{N/2}$ as equal to the corresponding matrix $\bar{A}_m(x)$ along either of its component rays outside of some circle $|x| = r$, within which they are chosen so as to satisfy the conditions of the preliminary theorem. Since $\bar{A}_1(x), \ldots, \bar{A}_N(x)$ are of the nature above described at $x = \infty$, the matrices $A_1(x), \ldots, A_{N/2}(x)$ are unlimitedly differentiable, and satisfy the permutability condition (see (21)) of the preliminary theorem. It is therefore possible to make such a choice of $A_1(x), \ldots A_{N/2}(x)$.[22]

It follows by the theorem that there exists a matrix $\Phi(x)$ of determinant nowhere zero save possibly at $x = a = 0$, analytic except along the rays arg $x = \tau_m'$ and such that

$$\lim_{x=x_m^+} \Phi(x) = [\lim_{x=x_m^-} \Phi(x)]\, \bar{A}_m(x_m), \tag{42}$$

where x_m is a point of arg $x = \tau_m'$ for which $|x_m| \geqq r$. It also follows from the theorem that each element of $\Phi(x)$ is represented asymptotically by a series in negative powers of x in each sector (τ'_m, τ'_{m+1}); since $A_m(x) \backsim I$, these series are the same in all the sectors and we may write

$$\Phi(x) \backsim (s_{ij}(x)), \tag{43}$$

where $s_{ij}(x)$ is a series of negative powers of x in which the determinant of the constant terms is not zero.

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[22] Cf. § 9.

Now write for $m = 1, \ldots, N+1$

$$Z_m(x) = \Phi(x)\, T(x) \qquad (\tau'_m \leqq \arg x \leqq \tau'_{m+1}).$$

From (41) and (42) there results along $\arg x = \tau'_{m+1}$

$$Z_{m+1}(x) = Z_m(x)\,[I + C_m], \qquad |x| \geqq r.$$

The functions $Z_m(x)$ may accordingly be continued analytically across each ray $\arg x = T_m$ and represent matrices analytic for $|x| \geqq r$, of determinant not zero. The relation (43) leads to the conclusion that

$$Z_m(x) \backsim S(x) \qquad (\tau'_m \leqq \arg x \leqq \tau'_{m+1}).$$

where $S(x)$ is of the desired form (35), and thence to the conclusion that this asymptotic representation is valid for $\tau_m \leqq \arg x < \tau_{m+1}$. The relation between $Z_{N+1}(x)$ and $Z_1(x)$ is that stated in (38).

In § 13 it was shown that, if matrices $Z_1(x), \ldots, Z_N(x)$ had the above properties and the further properties that they were analytic in the finite plane of determinant not zero for $x \neq 0$, and at $x = 0$ were properly equivalent to a Cauchy matrix, then these matrices were solutions of a canonical system (34) with irregular singular point at $x = \infty$. The same arguments can be used to establish that the $Z_1(x), \ldots, Z_N(x)$ before us are solutions of a differential system (24) having coefficients rational in character at $x = \infty$, with poles of order not more than p. But we have proved in § 12 that the matrix solution of such an equation is properly equivalent to that of a canonical linear differential system at $x = \infty$. Consequently a transformation $\overline{Z}(x) = A(x) Z(x)$, where $A(x)$ is a matrix of functions analytic in character at $x = \infty$ of determinant not zero there, makes $\overline{Z}(x)$ the solution of such a canonical system. In particular the matrices

$$\overline{Z}_1(x) = A(x)\, Z_1(x), \qquad \ldots\ldots, \qquad \overline{Z}_N(x) = A(x)\, Z_N(x)$$

form the solution of our problem, a fact which is apparent if we note that $\overline{S}(x) = A(x)\, S(x)$ has the same form as $S(x)$.

It has therefore been completely established under the stated restrictions that the characteristic constants which occur in the characterization of the matrix solutions of a canonical linear differential system can be chosen at pleasure.

The restriction that no three of the quantities $\alpha_1, \ldots, \alpha_n$ shall lie on a straight line is not essential, for if it is not satisfied and *if no two of the polynomials* $p_1(x), \ldots, p_n(x)$ *are identical* it will be possible to replace the rays $\tau_1, \ldots, \tau_N$ which have coalesced by an equal number

of curved rays so chosen that the real part of only one of the differences $p_i(x) - p_j(x)$ changes sign along the ray, and thence to apply the preliminary theorem in much the same fashion as before. If two or more polynomials $p_1(x), \ldots, p_n(x)$ are identical it is not necessary to modify the nature of the rays.

Furthermore it would be possible to construct an analogous existence proof when $S(x)$ has for its elements so-called *anormal series*.

In other words, the results here obtained are of an entirely general nature.

§ 15. *The Generalized Riemann Problem.*

The problem which I proposed in my paper on singular points (I) was the following: "*To construct a system of linear differential equations of the first order with prescribed singular points*

$$x_1, x_2, \ldots x_m, x_{m+1} = \infty$$

of respective ranks

$$q_1, \ldots, q_{m+1},$$

and with a given monodromic group, the characteristic constants being assigned for each singular point." This problem is virtually solved by what precedes.

It is of course understood that certain obvious conditions of compatibility are satisfied, the first being the one already noted,

$$T_{m+1} T_m \ldots T_1 = I.$$

However a second necessary condition must also be imposed. Take any assigned point $x = x_i$. The solution $Z_1(x)$ appertaining to this point (see § 14) is transformed successively into

$$Z_2(x)(1 + C_1)^{-1}, \quad Z_3(x)(I + C_2)^{-1}(I + C_1)^{-1}, \\ \ldots, Z_{N+1}(x)(I + C_N)^{-1} \ldots (I + C_1)^{-1},$$

as x passes over the rays $\tau_1, \ldots \tau_N$ respectively. Hence after a complete circuit of a_i, $Z_i(x)$ alters to

$$Z_1(x) I' (I + C_N)^{-1} \ldots (I + C_1),$$

where the matrix $\overline{T}_i$ of transformation is explicitly determined in terms of the characteristic constants. But in order for this set of characteristic constants to be possible, some solution $Z(x) = Z_1(x) C$ must undergo precisely the transformation by T_i, i. e.,

$$\overline{T}_i = C T_i C^{-1} \quad (i = 1, \ldots, m).$$

This second condition is satisfied if the elementary divisors associated with T_i and $\overline{T}_i$ are the same.

With the understanding that the two conditions of compatibility are satisfied, we can assert that there exist $m + 1$ canonical systems with the prescribed characteristic constants at $x_1, \ldots, x_{m+1}$ and with solutions $V_1(x), \ldots, V_{m+1}(x)$ undergoing a transformation to $V_1(x) T_1, \ldots, V_{m+1} T_{m+1}$ as x makes a positive circuit of $x_1, \ldots x_{m+1}$ respectively. Hence there will exist a matrix $Y(x)$ of elements analytic and of determinant not zero save at $x_1, \ldots, x_{m+1}$, properly equivalent to $V_i(x)$ at $x = a_i$ $(i = 1, \ldots, m)$, and properly or improperly equivalent to $V_{m+1}(x)$ at $x_{m+1} = \infty$. This follows upon application of the results of § 10. The matrix $Y(x)$ thus obtained is a matrix solution of a differential system of the form required and has the monodromic group and characteristic constants required, if the equivalence at $x_{m+1} = \infty$ be proper. This is an immediate consequence of the results of §§ 13, 14.

If this is not the case it is easy to show that if one merely increases the characteristic constants $r_1, \ldots, r_n$ for the singular point $x_{m+1} = \infty$ by suitable integers this equivalence at infinity becomes proper. For consider the matrix $S(x)$ corresponding to the matrix solution of the equivalent canonical system at $x = \infty$. The corresponding matrix $\overline{S}(x)$ for $Y(x)$ is of the form $A(x)\, S(x)$ where the elements of $A(x)$ are rational in character at $x = \infty$. Hence if a proper choice of C be made, the matrix $\overline{S}(x)\, C$ may be written (compare (33), § 11).

$$\begin{vmatrix} e^{p_1(x)} x^{\bar{r}_1}\left(a_{11} + \dfrac{b_{11}}{x} + \ldots\right), \ldots, e^{p_n(x)} x^{\bar{r}_n}\left(a_{1n} + \dfrac{b_{1n}}{x} + \ldots\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e^{p_1(x)} x^{\bar{r}_1}\left(a_{n1} + \dfrac{b_{n1}}{x} + \ldots\right), \ldots, e^{p_n(x)} x^{\bar{r}_n}\left(a_{nn} + \dfrac{b_{nn}}{x} + \ldots\right) \end{vmatrix}$$

where $\bar{r}_1, \ldots, \bar{r}_n$ differ from $r_1, \ldots, r_n$ by integers. By a process of reduction exactly like that given in § 11 this matrix may be replaced by a series of matrices of the same form in which the exponents $\bar{r}_1, \ldots, \bar{r}_n$ are increased so long as $|a_{ij}| \neq 0$. We conceive of $Y(x)$ as affected by the same series of operations, which have no effect on the properties above specified. Furthermore the transformed $Y(x)$ and $\overline{S}(x)$ stand always in the same relation to each other after as before transformation. If the process comes to an end so that $|a_{ij}| \neq 0$, the linear differential system with matrix solution $Y(x)$ will have a singular point of rank q_{m+1} at $x = \infty$ (compare § 13, pp. 547–548) and the

associated properly equivalent canonical system has characteristic constants only modified as stated.

Now we have

$$|Y(x)| \sim |S(x)|$$

in the *complete* vicinity of $x = \infty$, which is made up of the sectors $(\tau_1, \tau_2), \ldots (\tau_N, \tau_{N+1})$. It follows that the series for $|S(x)|$ converges and hence must be of the form

$$e^{p_1(x)+\ldots+p_n(x)} \cdot x^p \left(a + \frac{b}{x} + \ldots\right), \qquad a \neq 0,$$

so that always

$$r_1 + \ldots + r_n \leqq p.$$

But the above reductions increase $r_1 + \ldots + r_n$ and do not affect $|S(x)|$ or the value of p, and so must terminate.

We can state then that *either a solution of the stated problem, or of a modified problem in which the constants* $r_1, \ldots, r_n$ *of one of the singular points are altered to* $r_1 + l_1, \ldots, r_n + l_n$ *respectively, where* $l_., \ldots, l_n$ *are integers, will exist.*

The matrix $Y(x)$ thus obtained is not always unique. The most general determination is however of the form $P(x)\, Y(x)$ where $P(x)$ is a matrix of polynomials of constant determinant which fulfills other conditions. Thus the notion of "primitive systems" admits of extension to the case of irregular singular points.[23]

Part III; The Linear Difference Equation Problem.

§ 16. *Formulation of the Problem.*

Let

$$Y(x+1) = Q(x)\, Y(x) \tag{44}$$

be a linear difference system in which the elements of $Q(x)$ are polynomials of degree μ in x.[24] In my earlier paper on linear difference equations I demonstrated that, at least if the above equation admits a formal matrix solution

$$S(x) = [x^{\mu x} (\rho_j e^{-\mu})^x x^{r_j} s_{ij}(x)], \tag{45}$$

in which $s_{ij}(x)$ is a power series proceeding according to negative powers of x with the determinant of the leading coefficients not zero,

23 Cf. Plemelj, loc. cit. pp. 240–245. Like theorems may be proved in a similar manner here.

24 Essentially the most general linear difference system with rational coefficients may be reduced to this form; see II, § 5.

there exist two matrix solutions $Y^-(x)$ and $Y^+(x)$, with elements analytic in the finite plane save for poles, such that $Y^-(x) \backsim S(x)$ in any left half plane and $Y^+(x) \backsim S(x)$ in any right half plane.[25] The existence of such a solution was proved by Nörlund and Galbrun by methods based on the Laplace transformation somewhat earlier.[26]

These matrices $Y^-(x)$ and $Y^+(x)$ are connected by a relation

$$Y^-(x) = Y^+(x)\,P(x) \tag{46}$$

where $P(x)$ is evidently a matrix of periodic functions of period 1.

From the form of (44) it appears that $Y^-(x)$ is a matrix of entire functions, while $Y^+(x)$ is analytic save for poles.

In my paper I determined explicitly the nature of the elements $p_{ij}(x)$ of $P(x)$ to be the following:

$$\left\{\begin{aligned} p_{ii}(x) &= 1 + c_{ii}^{(1)}e^{2\pi\sqrt{-1}x} + \ldots + c_{ii}e^{2\pi(\mu-1)\sqrt{-1}x} + e^{2\pi r_i\sqrt{-1}}e^{2\pi\mu\sqrt{-1}x} \\ &\qquad\qquad (i = 1, \ldots, n) \\ p_{ij}(x) &= e^{2\pi\lambda_{ij}\sqrt{-1}x}[c_{ij}^{(0)} + \ldots + c_{ij}{}^{2\pi(\mu-1)\sqrt{-1}x}] \\ &\qquad\qquad (i \neq j;\ i, j = 1, \ldots, n), \end{aligned}\right. \tag{47}$$

Here λ_{ij} stands for the least integer as great as

$$\Re\left(\frac{1}{2\pi\sqrt{-1}}[\log\rho_j - \log\rho_i]\right). \tag{48}$$

An analogous determination in certain cases at about the same time was made by Nörlund (loc. cit.).

It is not difficult to show that, if $Y^-(x)$ and $Y^+(x)$ have the properties above outlined, then conversely they are solutions of a linear difference system (44) in which the elements of $Q(x)$ are rational if not polynomial.[27] For this reason the constants ρ_j, r_j, $c_{ij}^{(k)}$ may be called the characteristic constants of $Y^-(x)$ and $Y^+(x)$.

This characterization suggested to me the following problem: To construct a linear difference system (44) with assigned characteristic constants in which the elements of $Q(x)$ are polynomials in x of degree not greater than μ.

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25 These matrices $Y^-(x)$ and $Y^+(x)$ correspond to $G(x)$ and $H(x)$ of II. In certain cases it may be necessary to consider half planes not bounded by a vertical line. I refer to this possibility later.

26 Nörlund, Dissertation, Copenhagen (1911); Galbrun, Dissertation, Paris (1910).

27 See II, § 7.

§ 17. *Solution of the Problem of* § 16.

In order to treat the problem of § 16 we apply the preliminary theorem. We shall take $r = 1$, and take C_1 to be the axis of imaginaries in the complex plane unless $| P(x) | = 0$ at a point of that axis.

The matrix $A_1(x)$ is taken equal to

$$T(x)\, P(x)\, T^{-1}(x), \qquad T(x) = (x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j} \delta_{ij}),$$

except near to $x = 0$, where it is chosen in any way so as to satisfy the restrictions of the preliminary theorem there (compare § 9). Since the elements of $T(x)$ are in general multiple-valued functions of x, it is necessary to specify which branch of $T(x)$ to select. We shall choose a continuous branch of $T(x)$ in the right half plane, and a continuous branch in the left half plane in such a way that these branches coincide along the upper half of the axis of imaginaries. The first factor $T(x)$ in the expression for $A_1(x)$, will be identified with the first of these branches, and the last factor $T^{-1}(x)$ will be the inverse of the second of these branches.

It is therefore clear that, along the upper half of the axis of imaginaries, the element in the ith row and jth column of $A_1(x)$ is, for $i \neq j$,

$$e^{2\pi\lambda_{ij}\sqrt{-1}x} \rho_i^x \rho_j^{-x} x^{r_i - r_j} [c_{ij}^{(0)} + \ldots + c_{ij}^{(\mu-1)} e^{2\pi(\mu-1)\sqrt{-1}x}]$$

while the diagonal elements are the same as for $P(x)$. But by definition of λ_{ij},

$$1 > \Re\left(\lambda_{ij} - \frac{1}{2\pi\sqrt{-1}}(\log\rho_j - \log\rho_i)\right) \geqq 0. \tag{49}$$

Let us exclude at present the case of the equality sign; the element of $A_1(x)$ in the ith row and jth column ($i \neq j$) will therefore vanish to infinite order together with its derivatives as x goes to infinity along the upper half of the axis of imaginaries. The diagonal elements diminished by 1 have the same properties.

Hence we have $A_1(x) \backsim I$ along the upper half of the axis of imaginaries, while all the derivative matrices of $A_1(x)$ tend to matrices of zero elements as x becomes infinite. If $A_1(x)$ has this character along the lower half of the axis also, it is clear that this matrix satisfies all the restrictions imposed in the theorem.

Let us demonstrate that such is actually the case. The determination of $T(x)$ on the left-hand side of the lower half of the axis of imagi-

naries is obtained from that on the right-hand side by a complete positive circuit of $x = 0$, during which $T(x)$ changes from

$$(x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\delta_{ij}) \quad \text{to} \quad (e^{2\pi\mu\sqrt{-1}x}e^{2\pi\sqrt{-1}r_j}x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\delta_{ij}).$$

The ith diagonal element of $A_1(x)$ may now be written

$$c_{ii}^{(0)}e^{-2\pi\sqrt{-1}r_j}e^{-2\pi\mu\sqrt{-1}x} + \ldots + 1,$$

while the element in the ith row and jth column of $A(x)$ $(i \neq j)$ may be written

$$e^{2\pi(\lambda_{ij}-1)\sqrt{-1}x}\rho_i^x\rho_j^{-x}x^{r_i-r_j}\left[c_{ij}^{(0)}e^{-2\pi(\mu-1)\sqrt{-1}x} + \ldots + c_{ij}^{(\mu-1)}\right].$$

Bearing (49) in mind we readily perceive that $A_1(x)$ does have the indicated properties along the lower half of the axis of imaginaries.

According to the preliminary theorem we can then determine a matrix $\Phi(x)$ such that

$$(50) \qquad \lim_{x \doteq x_1-} \Phi(x) = \left[\lim_{x \doteq x_1+} \Phi(x)\right] A_1(x),$$

where x_1 is a point of the axis of imaginaries and the approach is from the left-hand and right-hand side of that axis respectively. If we take $x = a = 0$, the determinant of $\Phi(x)$ is not zero in the finite plane except at $x = 0$ possibly, and the elements of this matrix are analytic at any point not on the axis. Along the axis as defined from either side these elements have continuous derivatives of all order, and will be analytic at more than certain distance d from the origin. In the vicinity of $x = \infty$, $\Phi(x)$ is represented asymptotically by a matrix of series in $1/x$ with determinant of leading coefficients not zero. This matrix is the same on either side of the axis, since $A(x) \backsim I$.

Let us denote $\Phi(x)$ by $U^+(x)$ for x in the right half plane and by $U^-(x)$ for x in the left half plane, and write

$$\overline{Y}^+(x) = U^+(x)\,T(x), \qquad \overline{Y}^-(x) = U^-(x)\,T(x).$$

From equation (50) we see then that

$$(51) \qquad \overline{Y}^-(x) = \overline{Y}^+(x)\,P(x)$$

for $|x| > r$ along the axis of imaginaries. From the asymptotic form of $U^+(x)$ and $U^-(x)$ at $x = \infty$ we obtain

$$\overline{Y}^-(x) \backsim \overline{S}(x), \qquad \overline{Y}^+(x) \backsim \overline{S}(x),$$

in the left and right half plane, where $\overline{S}(x)$ is of the same form as $S(x)$ above. The relation (51) shows that $\overline{Y}^-(x)$ is composed of elements analytic in the right half plane.

Let us now apply the preliminary theorem a second time, taking $r = 1$, and for C_1 a circle with center at the origin and radius so large as to include within it all those points of the axis of imaginaries at which an element of $A_1(x)$ as chosen above is not analytic, and also so as not to pass through a zero of $|\,\overline{Y}^-(x)\,|$.

In this second application of the theorem we choose $A_1(x)$ to be $[\overline{Y}^-(x)]^{-1}$, and in this way satisfy the restrictions of the theorem. Furthermore let us take $a = \infty$.

Along the circle C_1 we have for the solution $\Phi(x)$

$$(52) \qquad \lim_{x=x_1^+} \Phi(x) = [\lim_{x=x_1^-} \Phi(x)]\,[\overline{Y}^-(x_1)]^{-1},$$

where the approach to the point x_1 of C is from without and within C respectively. Now write

$$Y^-(x) = \Phi(x)\,\overline{Y}^-(x), \qquad Y^+(x) = \Phi(x)\,\overline{Y}^+(x)$$

for x outside of C_1. It follows that $Y^-(x)$ is composed of elements analytic in this region; also along C_1, $Y^-(x)$ coincides with the inner determination of $\Phi(x)$ by (52), so that the elements of $Y^-(x)$ are also analytic within and on C_1. Hence $Y^-(x)$ is a matrix of entire functions. Similar considerations show that the elements of $Y^+(x)$ are analytic in the right half plane like the elements of $\overline{Y}^+(x)$.

At $x = \infty$, $Y^-(x)$ and $Y^+(x)$ are asymptotically represented by a matrix $S(x)$ in which however $r_1, \ldots, r_n$ are not necessarily the same as in $\overline{S}(x)$, and in which the determinant of the leading coefficients may be zero. This results from the fact that the elements of $\Phi(x)$ are rational in character at $x = \infty$ and in consequence can be expanded in convergent series in descending integral powers of x. Inasmuch as we have $|\,S(x)\,| = |\,\Phi(x)\,|\cdot|\,\overline{S}(x)\,|$, it is also true that $|\,S(x)\,|$ cannot reduce formally to zero.

Finally from (51) and (52) we infer that

$$(53) \qquad Y^-(x) = Y^+(x)\,P(x).$$

A first conclusion to be derived is that $Y^-(x) \backsim S(x)$ in any left half plane, and that also $Y^+(x) \backsim S(x)$ in any right half plane. In fact we have already determined the asymptotic form of $P(x)$, and this known form combined with the known asymptotic form of $Y^+(x)$ in the right half plane gives us the form of $Y^-(x)$ in the part of the plane to the left of any line parallel to the axis of imaginaries; a similar remark applies to the asymptotic form of $Y^+(x)$ in any right half plane.

One further remark comes in appropriately at this point. The definition of $\overline{Y}(x)$ and $\overline{Y}^{-}(x)$ ensured that $|\overline{Y}^{-}(x)|$ and $|\overline{Y}^{+}(x)|$ do not vanish to the left or right of the axis of imaginaries respectively, save possibly for $x = \infty$. Hence $|Y^{-}(x)|$ and $|Y^{+}(x)|$ do not vanish in these left or right half planes respectively.

We may now enter upon a series of modifications of $Y(x)$ which will preserve the above stated properties and secure in addition that the determinant of the leading coefficients of $S(x)$ is not zero. To this end we write the matrix $S(x)$ in the form

$$\begin{vmatrix} x^{\mu x}(\rho_1 e^{-\mu})^x x^{r_1}\left(a_{11}+\dfrac{b_{11}}{x}+\ldots\right), \ldots, x^{\mu x}(\rho_n e^{-\mu})^x x^{r_n}\left(a_{1n}+\dfrac{b_{1n}}{x}+\ldots\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x^{\mu x}(\rho_1 e^{-\mu})^x x^{r_1}\left(a_{n1}+\dfrac{b_{n1}}{x}+\ldots\right), \ldots, x^{\mu x}(\rho_n e^{-\mu})^x x^{r_n}\left(a_{nn}+\dfrac{b_{nn}}{x}+\ldots\right) \end{vmatrix}$$

and carry out reductions parallel to those given in § 11. The same reductions are supposed to be simultaneously effected upon $Y^{-}(x)$ and $Y^{+}(x)$ (compare § 15). This set of reductions will terminate, since $|S(x)|$ does not vanish identically, and, when it does, $Y^{-}(x)$ and $Y^{+}(x)$ will have the desired additional property.

Consider now the matrix

$$Q(x) = Y^{-}(x+1)[Y^{-}(x)]^{-1} = Y^{+}(x+1)[Y^{+}(x)]^{-1}.$$

These two forms for $Q(x)$ are equal in virtue of (53) and yield us at once the asymptotic form of the elements of $Q(x)$ in the complete vicinity of $x = \infty$ as descending power series in x with leading term of the μth degree at most.[28] Hence the elements of $Q(x)$ are rational in character at $x = \infty$, with a pole of at most order μ there.

In the finite plane to the left or along the axis of imaginaries the first expression for $Q(x)$ shows that the elements of $Q(x)$ are analytic without exception. It will be recalled that $Y^{-}(x)$ is a matrix of entire function of determinant not zero in the left half plane inclusive of the axis of imaginaries. On the other hand the elements of $Q(x)$ are analytic to the right of the axis of imaginaries, as the second form shows.

Accordingly $Q(x)$ is a matrix of polynomials of degree at most μ and $Y^{-}(x)$, $Y^{+}(x)$ are solutions of the rational difference system (44).

Finally it may be observed that in case $|P(x)| = 0$ along the axis of imaginaries, a parallel line may be used to take the same rôle; or

[28] Cf. I, § 7.

indeed any simple analytic curve without a horizontal tangent and with vertical asymptote, provided that $|P(x)| \neq 0$ along the curve.

If the equality sign obtains in (49) it will be necessary to employ a curve with asymptote not quite in the vertical and to employ half planes not bounded by a vertical line.

It is also possible to replace $S(x)$ by certain *anormal forms,*[29] and thus extend the above results to the most general case.

Our conclusion may be summed up as follows: *There exists a linear difference system* (44) *with matrix solutions* $Y^{-}(x)$, $Y^{+}(x)$ *which either possesses prescribed characteristic constants* $\rho_j, r_j, c_{ij}^{(k)}$, *or else constants* ρ_j, $r_j + l_j$, $c_{ij}^{(k)}$ *where* $l_1, \ldots, l_n$ *are integers. For an arbitrary curve which meets each line parallel to the real axis only once, having a vertical asymptote, and which does not pass through a point* $|P(x)| = 0$, *there exist such matrices* $Y^{-}(x)$, $Y^{+}(x)$ *with the further property that* $|Y^{-}(x)| \neq 0$ *to the left of the curve while the elements of* $Y^{+}(x)$ *are analytic and* $|Y^{+}(x)| \neq 0$ *to the right of the curve.*

It is worthy of note that this last property determines the location of the poles of the elements of $Y^{+}(x)$ completely: namely, they occur to the left of the curve and at the points for which $|P(x)| = 0$. This appears from the formula

$$Y^{+}(x) = Y^{-}(x)\, P^{-1}(x),$$

which also permits us to affirm that the precise maximum order of pole of any element of $Y^{+}(x)$ is the order of the zero of $|P(x)|$.

PART IV: THE LINEAR q-DIFFERENCE EQUATION PROBLEM.

§ 18. *On Linear q-Difference Equations.*

A linear q-difference system may be written

$$Y(qx) = Q(x)\, Y(x) \qquad |q| > 1, \tag{54}$$

where $Q(x)$ is a matrix of polynomials of degree μ or less, in analogy with the normal form (46) of linear difference systems. The apparently more general case in which the elements of $Q(x)$ are rational in x may be reduced to this form readily. Let the least common denominator of the elements of $Q(x)$, written as quotients of relatively prime polynomials, be

$$(x - a_1) \ldots (x - a_l)$$

29 Analogous to the anormal series for linear differential equations. These forms have recently been obtained by Mr. P. M. Batchelder.

and let $g_i(x)$ be a solution of the q-difference equation of the type (54)

$$g(qx) = (x - m)\,g(x) \tag{55}$$

for $m = a_i$. If one takes for new variable

$$\overline{Y}(x) = g_1(x)\ldots g_l(x)\,Y(x),$$

a new matrix equation (54) in $\overline{Y}(x)$ is obtained with $\overline{Q}(x)$ polynomial in x.

Let us write

$$t = \frac{\log x}{\log q}.$$

In terms of this new variable a solution of (55) for $m = 0$ is

$$q^{\frac{1}{2}(t^2-t)}.$$

For $m \neq 0$, the transformation

$$x = m\bar{x}, \qquad y(x) = e^{\pi\sqrt{-1}\frac{\log \bar{x}}{\log q}}\, m^{\frac{\log \bar{x}}{\log q}}\, \bar{y}(\bar{x})$$

takes (55) to the normal form

$$y(qx) = (1 - x)\,y(x). \tag{56}$$

Two solutions of this equation are

$$\left\{\begin{aligned} y_0(x) &= \left(1 - \frac{x}{q}\right)\left(1 - \frac{x}{q^2}\right)\ldots, \\ y_\infty(x) &= q^{\frac{1}{2}(t^2-t)}e^{-\pi\sqrt{-1}\,t}\,\frac{1}{1-\dfrac{1}{x}}\cdot\frac{1}{1-\dfrac{1}{qx}}\ldots, \end{aligned}\right. \tag{57}$$

as one may verify by direct substitution. The function $y_\infty(x)$ plays the same rôle for the linear q-difference equations as the gamma function does in the theory of linear difference equations. I have mentioned these functions in order to supply an example later.

The fundamental existence theorems for linear q-difference equations are essentially a consequence of the work of Grévy[30] and Leau.[31] The first complete treatment has been given by Carmichael,[32] and the

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30 Paris thesis, 1894.

31 Paris thesis, 1897.

32 Am. Jour. Math., **34**, 147–168 (1912).

result may be expressed as follows: There exist in general two matrix solutions

$$
\begin{cases} Y_0(x) = (x^{\rho_j} a_{ij}(x)) \\ Y_\infty(x) = q^{\frac{\mu}{2}(t^2-t)} (x^{-\sigma_j} b_{ij}(x)), \end{cases} \tag{58}
$$

where each function $a_{ij}(x)$ is analytic at $x = 0$, and each function $b_{ij}(x)$ is analytic at $x = \infty$; and where furthermore the determinants of the leading coefficients of $a_{ij}(x)$ and $b_{ij}(x)$ at $x = 0$ and $x = \infty$ respectively are not zero. It is only the case when such series exist that will be here considered.

It follows at once from (54) that $Y_0(x)$ is a matrix of functions analytic for $x \neq 0, \infty$, and that $Y^-(x)$ is a matrix of functions analytic in the finite plane except for poles when $x \neq 0$. Further, if we write

$$Y_0(x) = Y_\infty(x) P(x),$$

then $P(x)$ is a matrix of functions analytic for $x \neq 0, \infty$, and possessing the property that $P(qx) = P(x)$.[33] These properties are in close analogy with the properties for a linear difference system, to which indeed (54) reduces formally by the substitution $x = q^t$.

I propose now to determine completely the nature of $P(x)$, as I have done for the analogous functions $P(x)$ associated with the linear difference system; it is the doubly periodic functions which enter here instead of the simply periodic functions. By means of this determination it will be possible for us to state the problem which, for this field, is analogous to the problems above treated for linear differential and linear difference equations.

§ 19. *On the Matrix $P(x)$.*

Let us make the transformation $x = q^t$ and write

$$P(x) = \overline{P}(t).$$

The function $\overline{P}(t)$ is a single-valued function of t analytic save for poles; for, this transformation takes the Riemann surface of infinitely many leaves, with logarithmic branch points at $x = 0$ and $x = \infty$ in a one-to-one and conformal manner into the t-plane.

Let us conceive of the t-plane as divided into parallelograms which

33 Cf. Carmichael, loc. cit., p. 159.

belong to the periods $\omega = 1$, $\omega' = 2\pi\sqrt{-1}/\log q$, and let $ABCD$ be a parallelogram with vertices

$$x_0, \quad x_0 + 1, \quad x_0 + 1 + \frac{2\pi\sqrt{-1}}{\log q}, \quad x_0 + \frac{2\pi\sqrt{-1}}{\log q},$$

respectively. At homologous points of BC and AD, $\bar{P}(t)$ has the same value, since $\bar{P}(t+1) = \bar{P}(t)$.

To obtain the relation between $\bar{P}(t)$ at homologous points of AB and DC we consider first the matrix

$$(59) \qquad P(x) = Y_\infty^{-1}(x)\, Y_0(x).$$

It is apparent from the form of the elements of $Y_0(x)$ near $x = 0$ as given by (58) that, if a positive circuit of $x = 0$ be made, $Y_0(x)$ will change to $Y_0(x)K$, where

$$K = (e^{2\pi\rho_j\sqrt{-1}}\delta_{ij});$$

likewise, upon a similar circuit, $Y_\infty(x)$ will change to

$$(-1)^\mu e^{2\pi\mu\sqrt{-1}\,t} e^{-\frac{2\pi^2\mu}{\log q}}\, Y_\infty(x)\, L,$$

where

$$L = (e^{-2\pi\sigma_j\sqrt{-1}}\delta_{ij}).$$

If these modified matrices be substituted in (59) we obtain the form which $P(x)$ assumes after x has made a positive circuit of the origin. This is

$$(-1)^\mu e^{-2\pi\mu\sqrt{-1}\,t} e^{\frac{2\pi^2\mu}{\log q}} L^{-1} P(x)\, K.$$

If therefore $p_{ij}(x)$ denotes the element in the ith row and jth column of $P(x)$, such a circuit modifies $p_{ij}(x)$ to

$$(-1)^\mu e^{-2\pi\mu\sqrt{-1}\,t} e^{\frac{2\pi^2\mu}{\log q}} e^{2\pi(\sigma_i+\rho_j)\sqrt{-1}}\, p_{ij}(x).$$

But this circuit in the x-plane corresponds to a passage from a point of AB in the t-plane to the homologous point of DC; in this way, letting $\bar{p}_{ij}(t)$ stand for the element of $\bar{P}(t)$ in the ith row and jth column, we find

$$(60) \quad \bar{p}_{ij}\left(t + \frac{2\pi\sqrt{-1}}{\log q}\right) = (-1)^\mu e^{-2\pi\mu\sqrt{-1}\,t} e^{\frac{2\pi^2\mu}{\log q}} e^{2\pi(\sigma_i+\rho_j)\sqrt{-1}}\ \bar{p}_{ij}(t)$$

We have also seen that

$$\bar{p}_{ij}(t+1) = \bar{p}_{ij}(t). \tag{61}$$

Now let us attempt to satisfy (60) and (61) by writing

$$\bar{p}_{ij}(t) = ce^{at^2+bt}\sigma(t-a_1)\ldots\sigma(t-a_\mu), \tag{62}$$

where $\sigma(t)$ is the Weierstrass sigma function belonging to the periods

$$\omega = 1, \qquad \omega' = \frac{2\pi\sqrt{-1}}{\log q},$$

which satisfies the relations

$$\left\{\begin{aligned} &\sigma\left(t+\frac{2\pi\sqrt{-1}}{\log q}\right) = -e^{\eta'\left(t+\frac{\pi\sqrt{-1}}{\log q}\right)}\sigma(t), \\ &\sigma(t+1) = -e^{\eta(t+\frac{1}{2})}\sigma(t), \qquad \eta\frac{2\pi\sqrt{-1}}{\log q} - \eta' = 2\pi\sqrt{-1}. \end{aligned}\right. \tag{63}$$

The above choice of ω and ω' meets the requirement $\Re(\omega'/\omega) > 0$, since $|q| > 1$.

A direct substitution into (61) determines

$$a = -\eta\frac{\mu}{2}, \qquad b = \eta\sum_{i=1}^{\mu}a_i + \mu\pi\sqrt{-1} + 2k\pi\sqrt{-1}, \tag{64}$$

in which k denotes any integer. If these values of a and b are taken and a direct substitution is made in (60), there is obtained the further condition

$$\sum_{i=1}^{\mu}a_i + \frac{\mu\pi\sqrt{-1}}{\log q} = \sigma_i + \rho_j + l - \frac{2k\pi\sqrt{-1}}{\log q}, \tag{65}$$

in which l denotes an integer. These conditions are necessary and sufficient that an expression of the form (62) shall satisfy both (60) and (61). If the value for $\sum_{i=1}^{\mu}a_i$ deduced from (65) be used in (64) the expression for b simplifies to

$$\eta(\sigma_i + \rho_j + l) - \eta'\left(\frac{\mu}{2} + k\right).$$

But if one adds or subtracts a period to a_i the precise effect is to alter k or l by an integer. It is therefore always permissible to take $k = l = 0$ and to write

$$\bar{p}_{ij}(t) = c_{ij}e^{\frac{-\eta\mu}{2}t^2+\left[\eta(\sigma_i+\rho_j)-\frac{\eta'\mu}{2}\right]t}\sigma(t-a_1^{(i,j)})\ldots\sigma(t-a_\mu^{(i,j)}), \tag{66}$$

provided that

$$\sum_{\lambda=1}^{\mu} \alpha_\lambda^{(i,j)} = \sigma_i + \rho_j - \frac{\mu\pi\sqrt{-1}}{\log q}. \tag{67}$$

This is of course under the assumption that $\bar{p}_{ij}(t)$ may be represented in the form (62). But this fact may be proved at once. For let $\psi(t)$ be any function which satisfies (60) and (61), and $\phi(t)$ the particular one above obtained. The function $\psi(t)/\phi(t)$ is doubly periodic, analytic save for poles, and can therefore be expressed as a quotient of products of sigma functions

$$C\frac{\sigma(t-\gamma_1)\ldots\sigma(t-\gamma_k)}{\sigma(t-\beta_1)\ldots\sigma(t-\beta_k)}, \qquad \Sigma\gamma_k = \Sigma\beta_k.$$

But this quotient when multiplied by $\phi(t)$, which is expressed as a product of sigma functions, must yield $\psi(t)$, an entire function. This necessitates that for each zero of $\sigma(t-\gamma_j)$ in the numerator $(i = 1, \ldots, \mu)$ there must be a congruent zero $x = \beta_j$ of $\sigma(x-\beta_j)$ in the denominator. Such pairs of corresponding factors may be combined leaving only an exponential factor e^{ct+d}. Thus $\psi(t)$ appears in the same form as $\phi(t)$.

Our result is therefore that the element $p_{ij}(x)$ of $P(x)$ is of the form (66) $(t = \log x/\log q)$, where $\rho_1, \ldots, \rho_n, \sigma_1, \ldots, \sigma_n$ are the constants that appear in the series representations (58), and where the conditions (67) are fulfilled.

It is interesting to apply these results to the equation (56) in which $n = 1$, $\mu = 1$, $\rho_1 = 0$, $\sigma_1 = \pi\sqrt{-1}/\log q$, and $Y_0(x)$ and $Y_\infty(x)$ reduce respectively to $y_0(x)$ and $y_\infty(x)$ defined in (57). In this case we have therefore

$$y_0(x) = y_\infty(x)\, p(x) \tag{68}$$

where

$$p(x) = c\, e^{\frac{-\eta}{2}\left(\frac{\log x}{\log q}\right)^2 - \pi\sqrt{-1}\frac{\log x}{\log q}}\,\sigma\left(\frac{\log x}{\log q}\right).$$

The constant c may be determined by writing (68) in the form

$$y_0(x) = [(x-1)\,y_\infty(x)]\left[\frac{p(x)}{x-1}\right]$$

and allowing x to approach 1. Since $\sigma(0) = 0, \sigma'(0) = 1$, this gives (see (57))

$$c = \left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{q^2}\right)^2\ldots \tag{69}$$

The relation (68) with the explicit values of $y_0(x)$, $y_\infty(x)$ and $p(x)$ substituted in is essentially one of the fundamental product formulas for the sigma function.

§ 20. *The q-Difference Equation Problem.*

It is now easy to show that conversely if $Y_0(x)$, $Y_\infty(x)$ are matrices of functions of the form (58) in the vicinity of $x = 0$ and $x = \infty$ respectively, analytic for $x \neq 0, \infty$, save for poles, and if the matrix $P(x)$, defined by the relation $Y_0(x) = Y_\infty(x)P(x)$, is composed of elements $p_{ij}(x)$ which are left unchanged when x is replaced by qx, then $Y_0(x)$ and $Y_\infty(x)$ are matrix solutions of a linear q-difference system (54) with rational coefficients. In fact, if we write

$$Q(x) = Y_0(qx)\,Y_0^{-1}(x) = Y_\infty(qx)\,Y_\infty^{-1}(x),$$

it is seen at once that for $x \neq 0, \infty$, the only singularities of $Q(x)$ are poles, while the first and second of these forms for $Q(x)$ ensure that $Q(x)$ is composed of elements analytic at $x = 0$ and with a pole of at most order μ at $x = \infty$ if not analytic there. This suffices to establish the fact that the elements of $Q(x)$ are rational.[34] In order to conclude further that the elements of $Q(x)$ are polynomials of degree μ it is sufficient to know that the plane may be divided into two parts by a loop about $x = 0$ meeting each equiangular spiral or radial line

$$\theta = c + \frac{\arg q}{\log|q|}\log r \qquad (r, \theta, \text{ polar coördinates}) \tag{70}$$

only once and not passing through a point $|P(x)| = 0$, such that the elements of $Y_0(x)$ are analytic and $|Y_0(x)|$ is not zero within or along the loop, while the elements of $Y_\infty(x)$ are analytic and $|Y_\infty(x)|$ is not zero outside the loop. Under these conditions the first expression for $Q(x)$ makes it evident that its elements are analytic within or along the loop, and the second expression makes it clear that the same is true without the loop, since if x is a point without the loop so is qx.

It is natural to term the $2n$ constants ρ_j, σ_j and the $n^2(\mu+1)$ constants c_{ij}, $a_1^{(i,j)}, \ldots, a_\mu^{(i,j)}$ *the characteristic constants.* These constants are not all independent, since there are n^2 relations between the constants $a_k^{(i,j)}$ of the type (67). Furthermore the constants c_{ij} are not uniquely determined by the given q-difference system. For, any ith column of $Y_0(x)$ is only determined up to a constant factor f_i and like-

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34 Compare II, § 7.

wise any ith column of $Y_\infty(x)$ is only determined up to a constant factor h_i. This fact appears from (58). The effect is to allow one to replace c_{ij} in $p_{ij}(x)$ by $f_j c_{ij}/h_i$, and thus vary at will $2n - 1$ of the characteristic constants c_{ij}. There remain then essentially only

$$2n + n^2(\mu + 1) - n^2 - (2n - 1) \quad \text{or} \quad n^2\mu + 1$$

characteristic constants. It is readily verified that these constants are all invariant under a transformation $\overline{Y}(x) = C\overline{Y}(x)$, where C is an arbitrary matrix of constants.

But the equation (54) involves $n^2(\mu + 1)$ arbitrary coefficients in $Q(x)$ of which there are $n^2(\mu + 1) - (n^2 - 1)$ or $n^2\mu + 1$ invariants under the same linear transformation. Hence we have found as many invariants for the q-difference system as invariant characteristic numbers for the solutions.

We are thus led to formulate the following problem: To construct a q-difference system (54), with coefficients of degree not greater than μ in x, having any assigned set of characteristic constants.

§ 21. *Solution of the Problem of* § 20.

Here also we shall employ the preliminary theorem, but the application of it is even simpler than in the earlier cases.

The conclusion that we shall derive is the following: *There exists a linear q-difference system* (54) *with the matrix solutions* $Y_0(x)$, $Y_\infty(x)$ *either possessing prescribed characteristic constants* $\rho_i, \sigma_i, c_{ij}, a_1^{(i,j)}, \ldots, a_\mu^{(i,j)}$ *or else constants* ρ_j, $c_j + l_j$, c_{ij}, $a_1^{(i,j)}, \ldots, a_\mu^{(i,j)}$, *where* $l_1, \ldots, l_n$ *are integers. For an arbitrary loop about* $x = 0$ *which cuts each spiral* (70) *only once and does not pass through a point* $|P(x)| = 0$, *there exist matrices* $Y_0(x)$, $Y_\infty(x)$ *with the further property that* $|Y_0(x)| \neq 0$ *within or along the loop while the elements of* $Y_\infty(x)$ *are analytic and* $|Y_\infty(x)|$ *is not zero without the loop.*

Let C_1 be a specified loop of this description which may be taken to be analytic.[36] We may take the matrix $A_1(x)$ of the preliminary theorem to be

$$T_\infty(x)\, P(x)\, T_0^{-1}(x),$$

where

$$T_0(x) = (x^{\rho_j}\delta_{ij}), \qquad T_\infty(x) = q^{\frac{\mu}{2}(t^2-t)}\,(x^{-\sigma_j}\delta_{ij}),$$

since the elements of $A_1(x)$ are single-valued and analytic along C_1 by (60).

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36 It is only a question as to how the loop weaves among the zeros of $|P(x)|$.

According to the theorem there exists a matrix $\Phi(x)$ such that

$$\lim_{x=x_1^+} \Phi(x) = [\lim_{x=x_1^-} \Phi(x)]\, A_1(x_1), \tag{71}$$

where x_1 is an arbitrary point of C_1 and the $+$ and $-$ signs denote approach to x_1 from within and without C_1 respectively; the matrix $\Phi(x)$ possesses certain other properties: it is composed of elements analytic for x not on C_1, save at a point a which we shall take to be at infinity; furthermore its determinant does not vanish in the finite plane.

Let us denote by $U_0(x)$ the matrix $\Phi(x)$ within C_1, and its analytic extension across C_1; similarly by $U_\infty(x)$ let us denote the matrix $\Phi(x)$ without C_1 and its analytic extension across C_1. Consider then the matrices

$$Y_0^-(x) = U_0(x)\, T_0(x), \qquad Y_\infty(x) = U_\infty(x)\, T_\infty(x).$$

From (71) we obtain at once

$$Y_0^-(x) = Y_\infty(x)\, P(x), \tag{72}$$

and prove without difficulty that $Y_0^-(x)$ and $Y_\infty^+(x)$ as thus defined have the characteristics demanded save possibly that $Y_\infty(x)$ may not be precisely of the form (58) at $x = \infty$, as it would be if $\Phi(x)$ were composed of elements analytic at $x = \infty$ and if also $|\Phi(x)|$ were different from zero at $x = \infty$. Nevertheless one can always write $Y_\infty(x)$ in the form

$$q^{\frac{1}{2}(t^2-t)} \left| \begin{matrix} x^{-\bar{\sigma}_1}\left(a_{11} + \frac{b_{11}}{x} + \dots\right), & \dots & x^{-\bar{\sigma}_n}\left(a_{1n} + \frac{b_{1n}}{x} + \dots\right) \\ x^{-\bar{\sigma}_1}\left(a_{n1} + \frac{b_{n1}}{x} + \dots\right), & \dots & x^{-\bar{\sigma}_n}\left(a_{nn} + \frac{b_{nn}}{x} + \dots\right) \end{matrix} \right|$$

where $\bar{\sigma}_1, \dots, \bar{\sigma}_n$ differ from $\sigma_1, \dots, \sigma_n$ by integers. By a process of reduction precisely like that employed earlier (§ 11) one may further modify $Y_0(x)$ and simultaneously $Y_\infty(x)$ so as to preserve all of the properties already noted and to finally obtain $|a_{ij}| \neq 0$. It is to be recalled that $|Y_0(x)|$ is not identically zero and can at most vanish to a finite order at $x = \infty$; for it is this fact that enables us to conclude that the process of reduction terminates.

The argument of the preceding paragraph shows that $Y_0(x)$ and $Y_\infty(x)$ will be matrix solutions of a system (54) with coefficients polynomials in x of degree μ at most.

It is deserving of notice that the properties stated above determine the location of the poles of the elements of $Y_\infty(x)$, namely at the zeros of $| P(x) |$ within the loop, the maximum order of the pole of any element being precisely the order of the corresponding zero of $| P(x) |$. This is an immediate consequence of the relation.

$$Y_\infty(x) = Y_0(x) P^{-1}(x).$$

Harvard University,
Cambridge, Mass.